Structural mechanics 2

Summer semester 2023/24

Structural mechanics 2

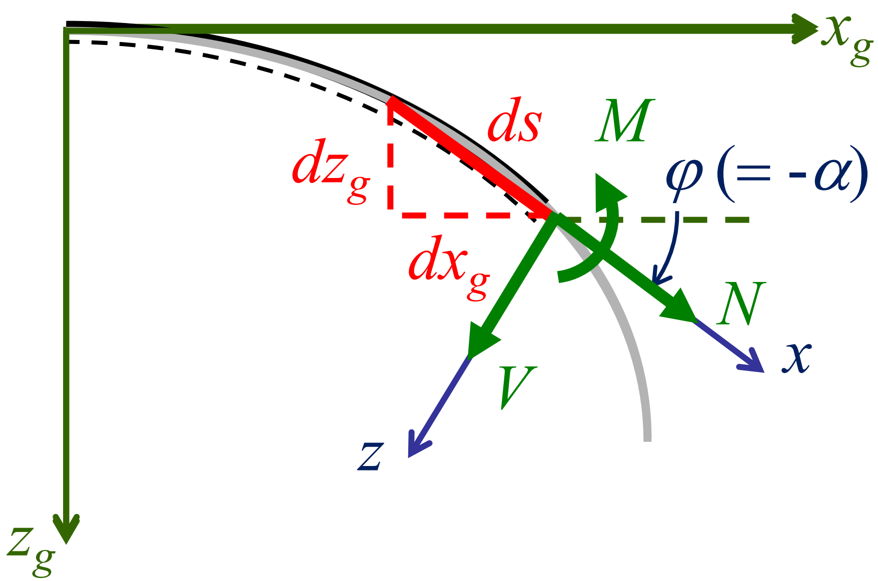
Lecture no. 7, April 2, 2024

1) Internal forces – summary
2) 3D beams

Review – lecture No. 5

Planar curved beam - geometry

Rotation of local CS:



Centerline is defined by the function:

$$z_g = g\left(x_g\right), \text{ Eg. } z_g = 3{x_g}^2$$

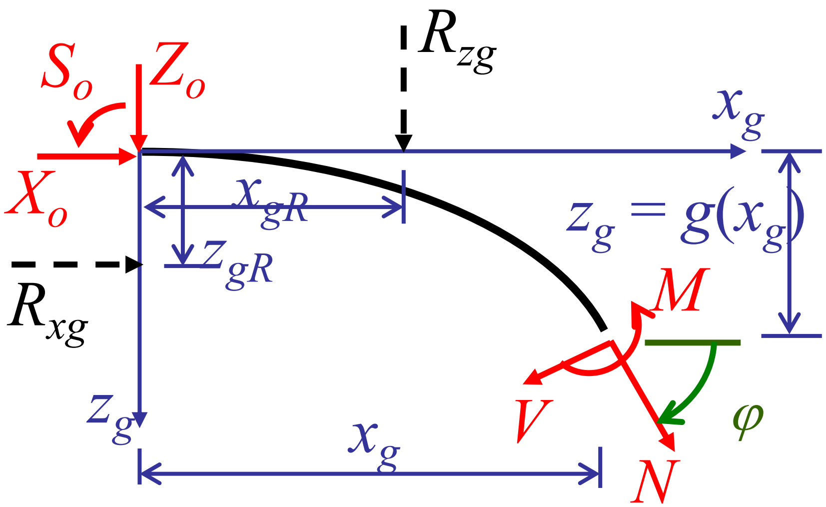
$$\frac{dz_g}{dx_g} = g' \Rightarrow dz_g = g' dx_g$$

$$ds = \sqrt{dx_g^2 + dz_g^2} = \sqrt{1+\left(g'\right)^2}\, dx_g$$

$$\cos(\varphi) = \frac{dx_g}{ds} \quad \Rightarrow \quad \cos\varphi = \frac{1}{\sqrt{1+\left(g'\right)^2}}$$

$$\sin(\varphi) = \frac{dz_g}{ds} \quad \Rightarrow \quad \sin\varphi = \frac{g'}{\sqrt{1+\left(g'\right)^2}}$$

Planar curved beam - internal forces



Internal forces (equilibrium)

$$N + X_0 \cos\varphi + Z_0 \sin\varphi + R_{xg} \cos\varphi + R_{zg} \sin\varphi = 0 \Rightarrow N\left(x_g\right) = \ldots$$

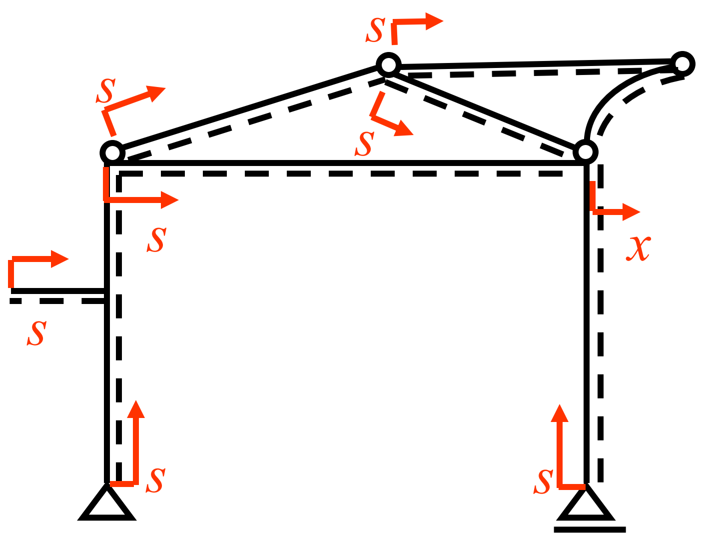
$$V - X_0 \sin\varphi + Z_0 \cos\varphi - R_{xg} \sin\varphi + R_{zg} \cos\varphi = 0 \Rightarrow V\left(x_g\right) = \ldots$$

$$M - X_0 \cdot g\left(x_g\right) + Z_0 \cdot x_g - R_{xg}\left(g\left(x_g\right) - z_{gR}\right) + R_{zg}\left(x_g - x_{gR}\right) + S_0 = 0 \Rightarrow M\left(x_g\right) = \ldots$$

$$\left(\cos\varphi = \frac{1}{\sqrt{1+\left(g'\right)^2}} \qquad \sin\varphi = \frac{g'}{\sqrt{1+\left(g'\right)^2}} \right)$$

Beam systems

System of straight, inclined, cranked and curved beams



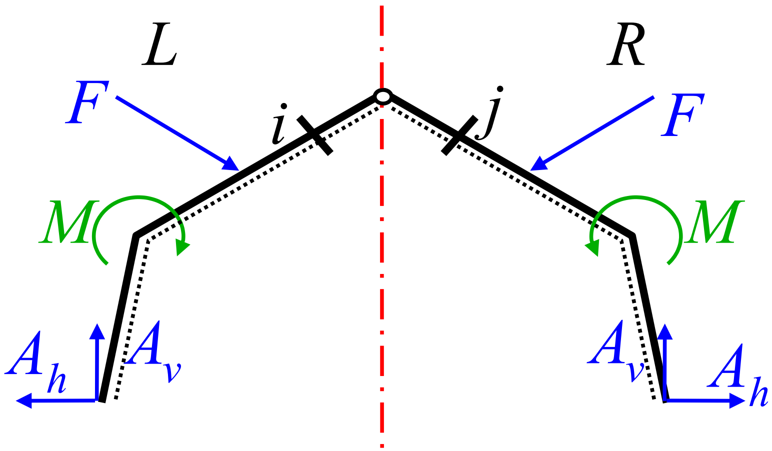
<u>Calculation of internal forces</u>

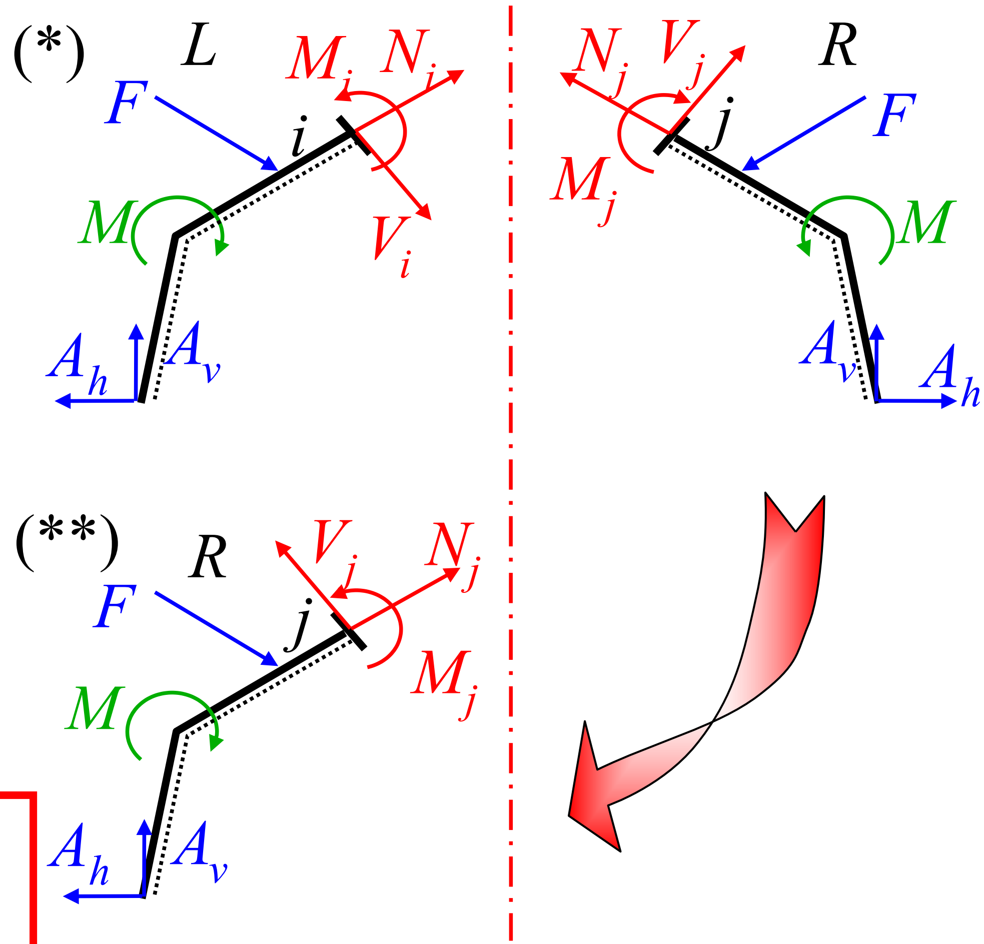
a) all internal and external reactions are determined

b) local coordinate system is defined in each beam

c) calculation of internal forces of individual beams

Symmetric structures - symmetric loads





Compare (*) and (**)

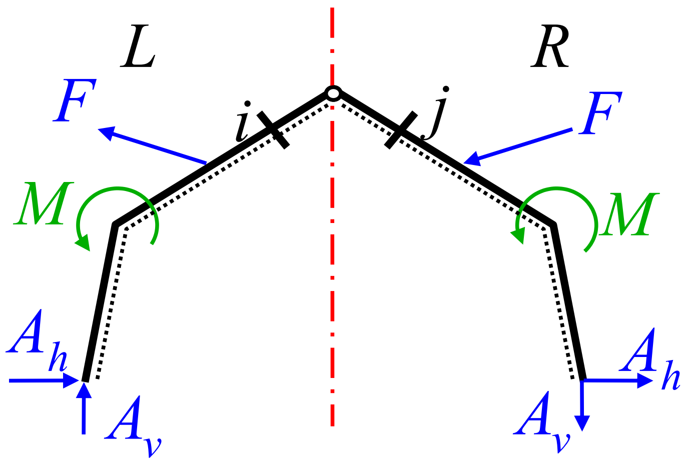
$$N_i = N_j$$

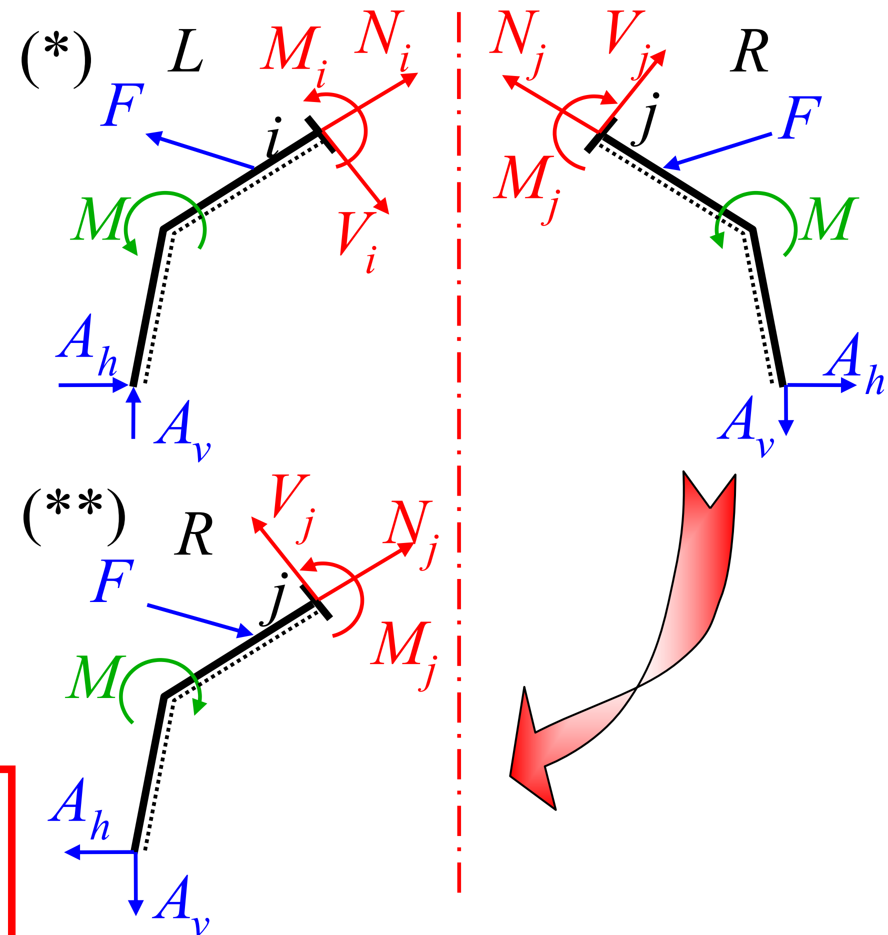
$$M_i = M_j$$

$$V_i = -V_j$$

- distributions of M and N are symmetric
- distribution of V is antisymmetric

Symmetric structures - antisymmetric loads





Compare (*) and (**)

$$N_i = -N_j$$

$$M_i = -M_j$$

$$V_i = V_j$$

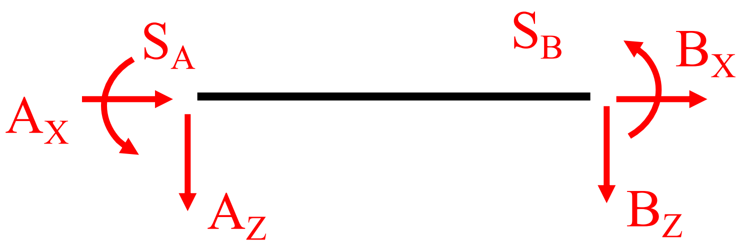
- distributions of M and N are antisymmetric
- distribution of V is symmetric

Lecture No. 6

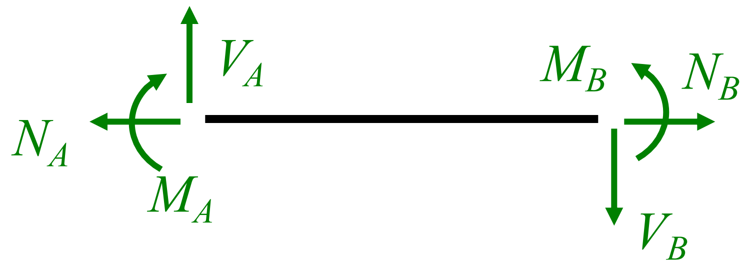
Result check

- equilibrium conditions / equivalency of forces in the end points

Reactions



Internal forces



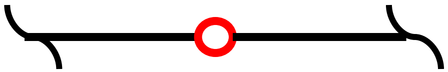
$$N_A = -A_X \quad , \quad V_A = -A_Z \quad , \quad M_A = -S_A$$

$$N_B = B_X \quad , \quad V_B = B_Z \quad , \quad M_B = S_B$$

Result check

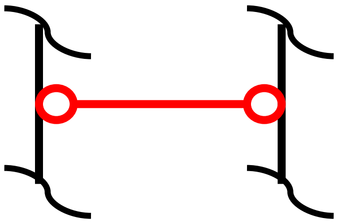
- internal forces in bonds/elements, eg.:

  - non-loaded hinge ($M_{ext} = 0$) => $\boldsymbol{M} = \mathbf{0}$



  - non-loaded truss element

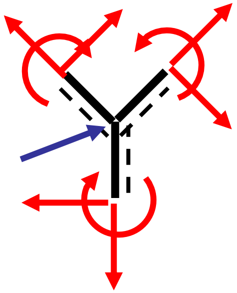
    => $\boldsymbol{V} = \boldsymbol{M} = \mathbf{0}$



  - non-loaded free end

    => $\boldsymbol{N} = \boldsymbol{V} = \boldsymbol{M} = \mathbf{0}$

- equilibrium conditions in joints



Result check

- differential relations between internal forces and the load

  … relations between load functions and internal forces

  … position of extremes

- determination of internal forces by means of independent method

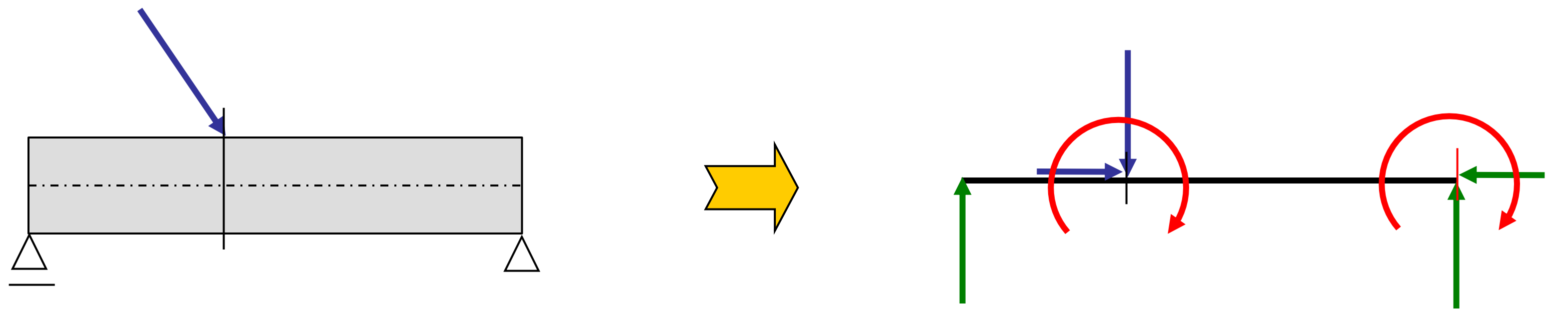
  - calculation from the “other end”
  - deformation method, etc.

- symmetry and antisymmetry

Internal forces
- summary

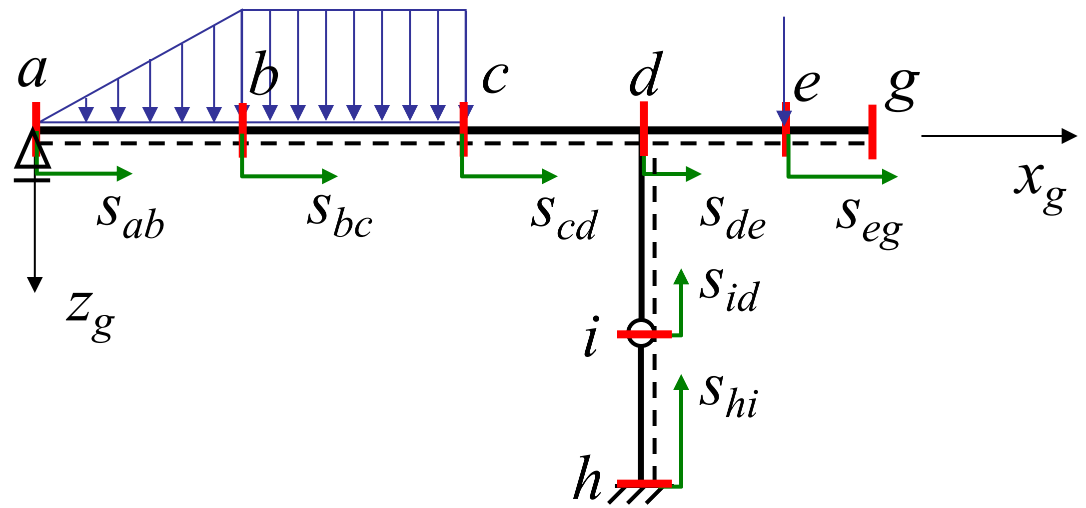
- we need to know the external forces (load and reactions)
  - all necessary reactions are evaluated

- internal forces are defined with respect to the centerline, all forces acting on the structure are transformed to the centerline



Internal forces - summary

- centerline is divided into intervals



- internal forces are defined as functions with respect to the local coordinate s for each interval: $N(s)$, $V(s)$, $M(s)$

  - equilibrium/equivalency approach or

  - solution of differential equations with boundary conditions

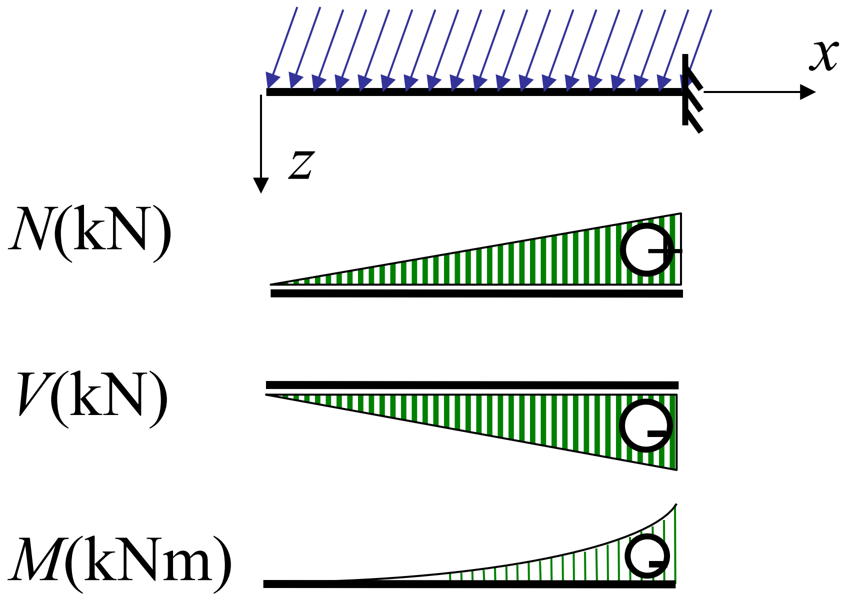
Internal forces
- summary

- determine the positions of extreme values in each interval

  - solution of equations: $\frac{dN}{ds} = 0, \quad \frac{dV}{ds} = 0, \quad \frac{dM}{ds} = 0$

  - boundary values of each interval

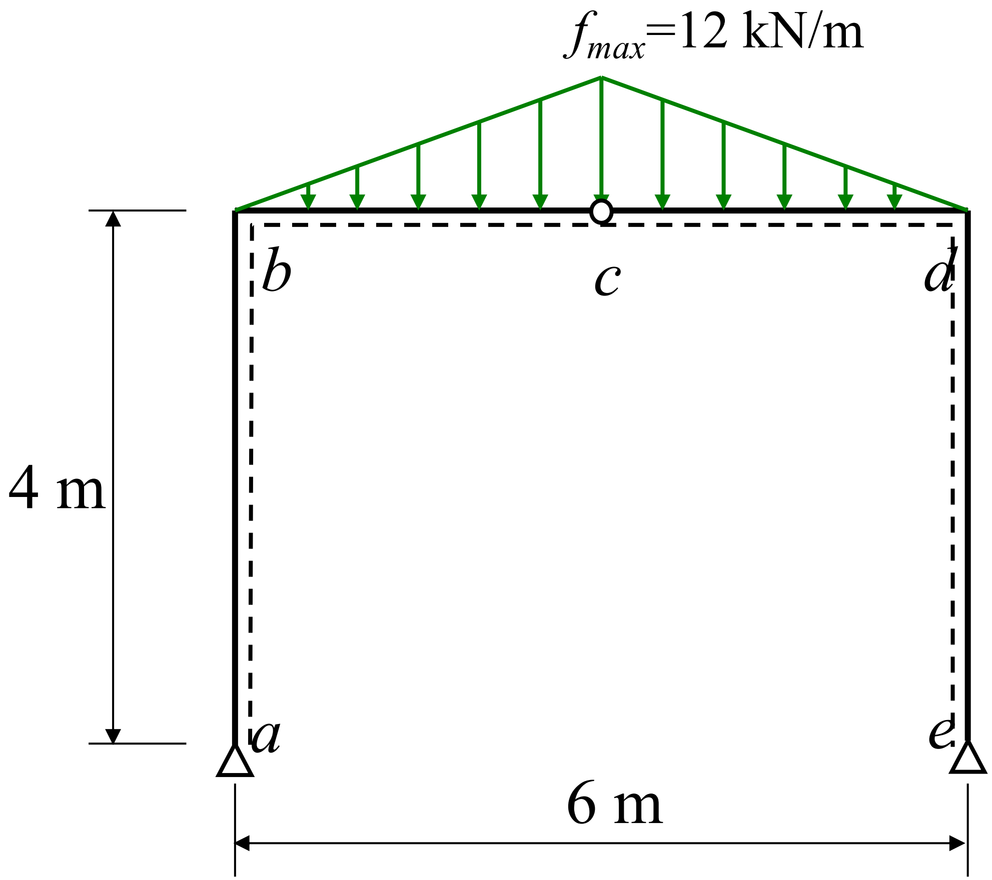
- plotting of internal forces



- check of results

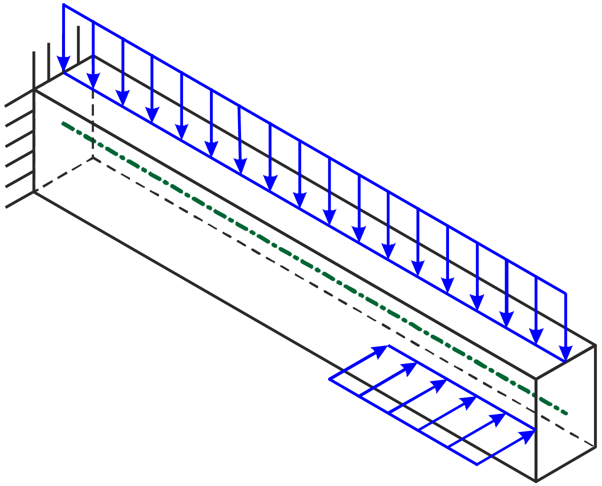
Internal forces - example

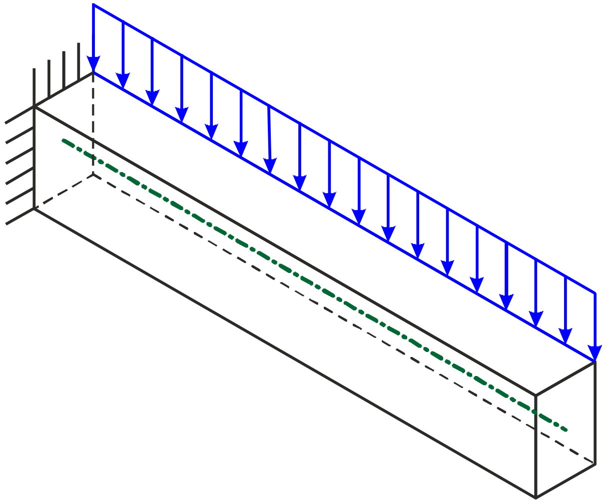
Draw the distributions of internal forces and their extreme values. Write the internal forces as a function of position coordinate for the interval (b, c).



3D beams

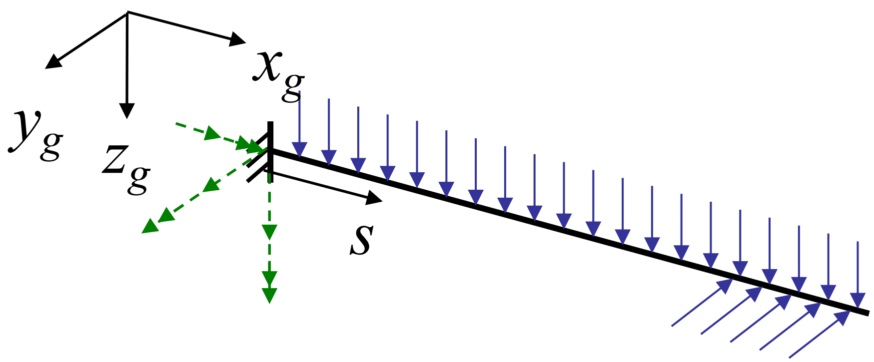
- Centerline: line segment
- Load and reactions:
  - Force vectors are not in the same plane
  - Force vectors are not in the centerline plane
- Eg.:

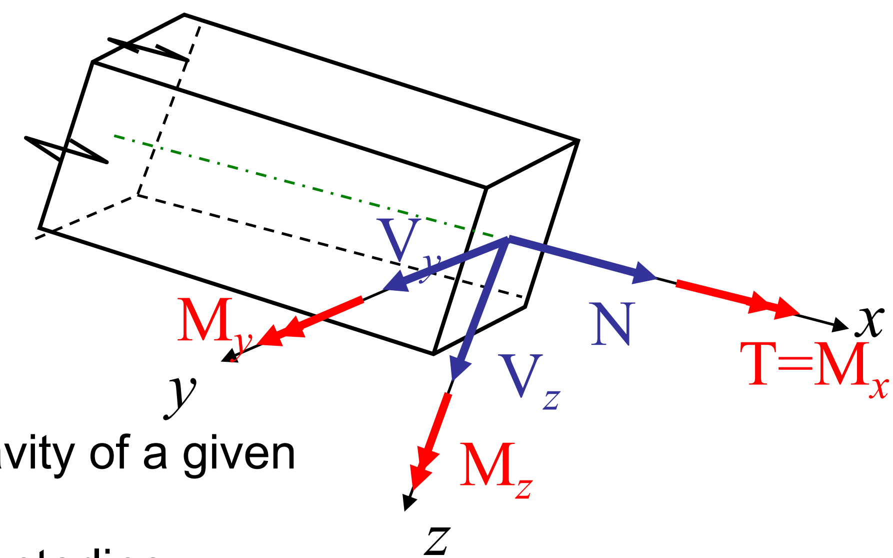




3D beams

- Coordinate systems
  - global x_g-y_g-z_g
  - local x-y-z (orientation and direction of internal forces)
  - local s (internal forces as a function of cross-section position)



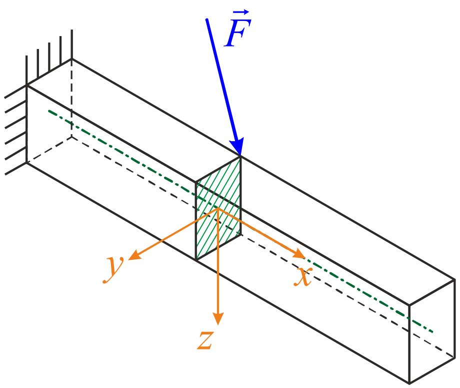


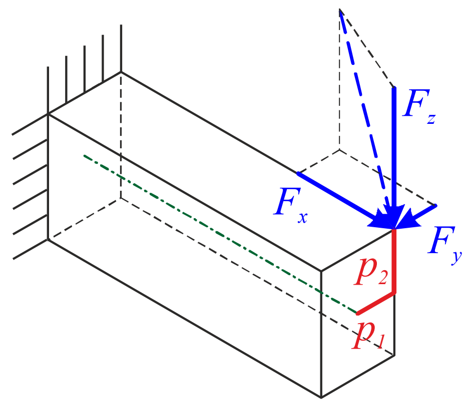
- Origin of *x-y-z* is in the center of gravity of a given cross-section
- Direction of *x-axis*: tangent to the centerline
- Directions of *y and z axes*:

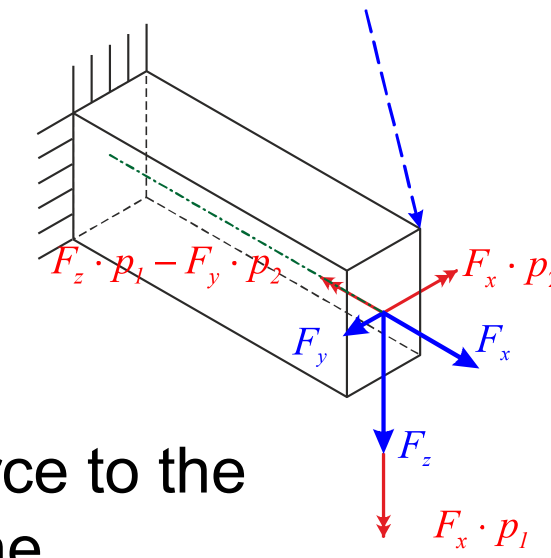
  symmetric cross-sections – axes of symmetry
  general cross-section – principal axes of inertia

3D beams – load transformation

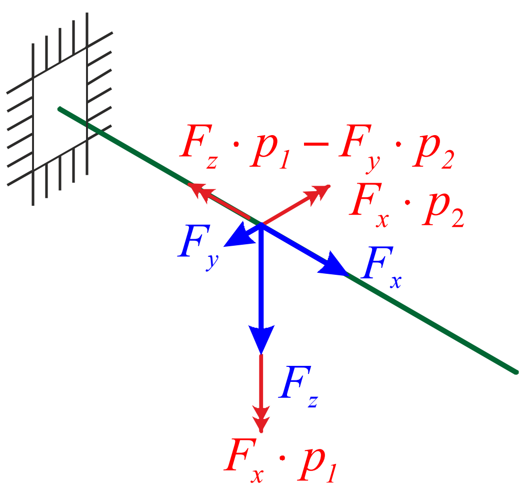
Point load







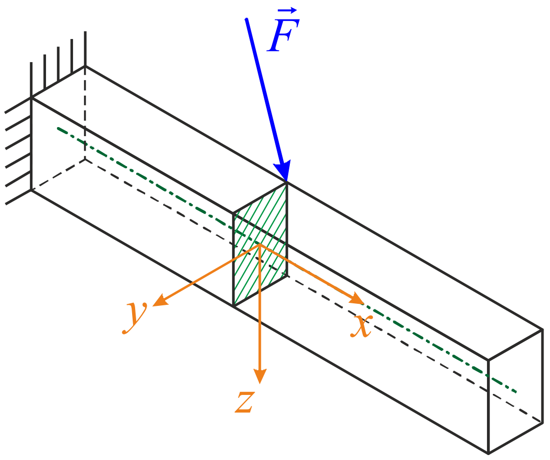
Model:

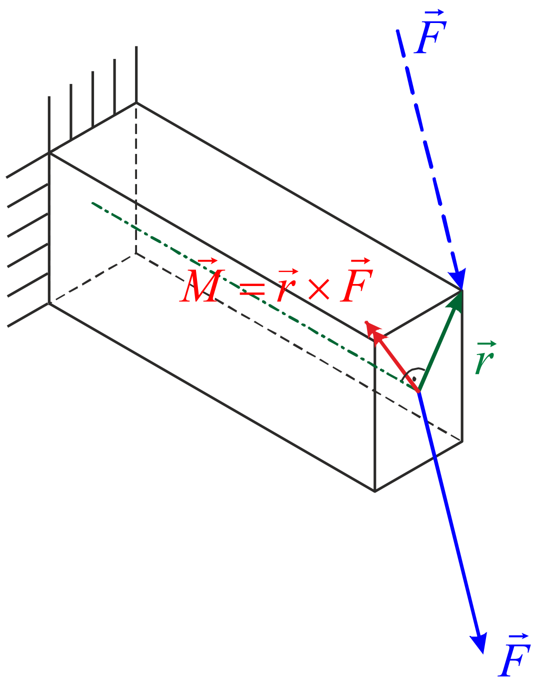


... Transformation of force to the center of gravity of the corresponding cross-section

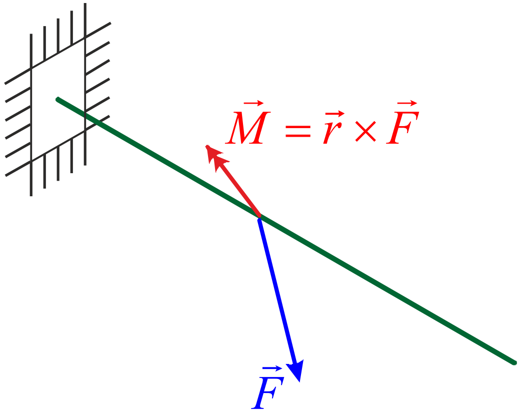
3D beams – load transformation

Point load - general approach





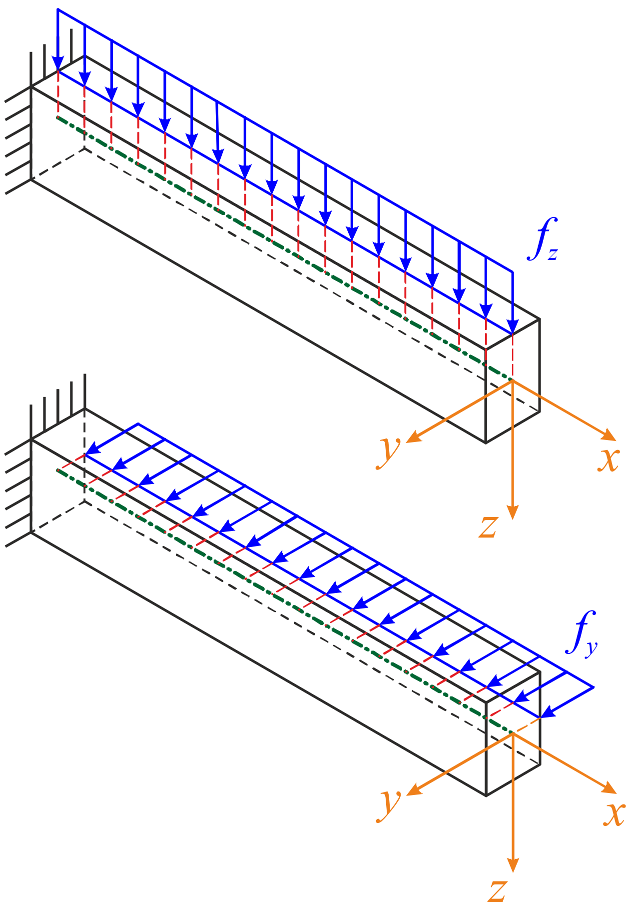
Model:



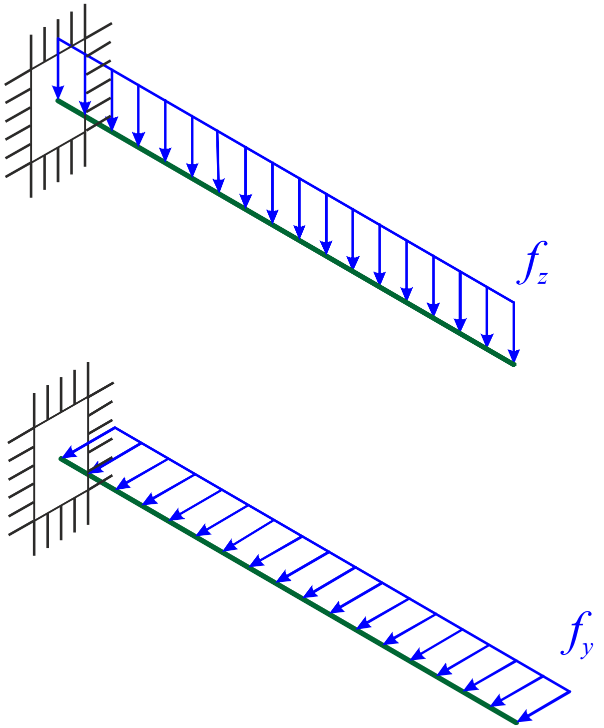
... Transformation of force to the center of gravity of the corresponding cross-section

3D beams – load transformation

Distributed load acting perpendicularly to the centerline

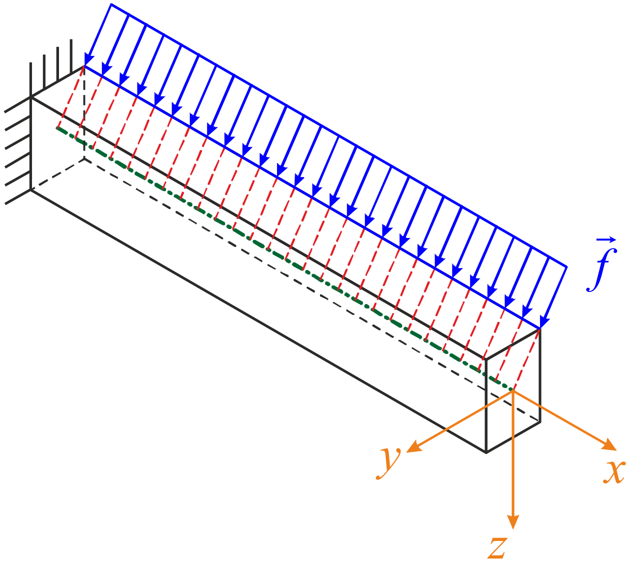


Model:

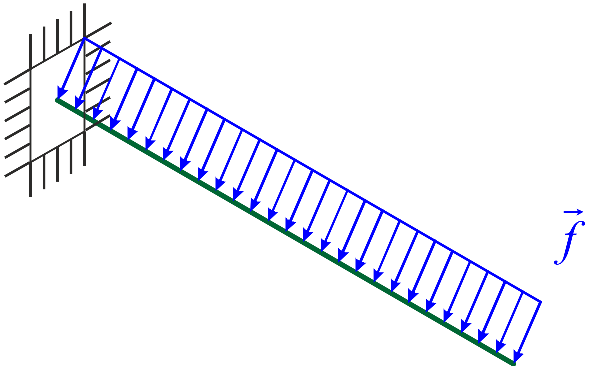


3D beams – load transformation

Distributed load acting perpendicularly to the centerline

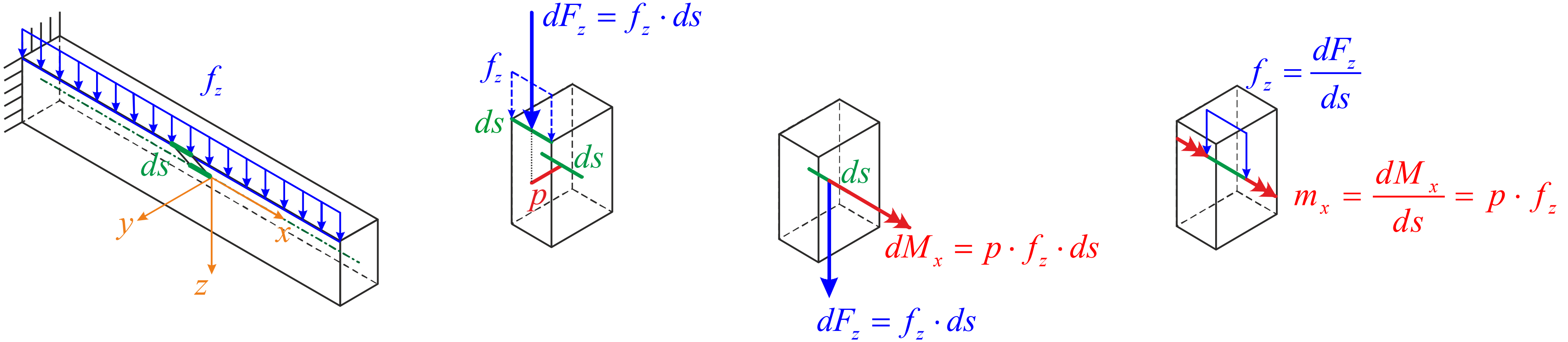


Model:



3D beams – load transformation

Distributed load acting on the line parallel to the centerline

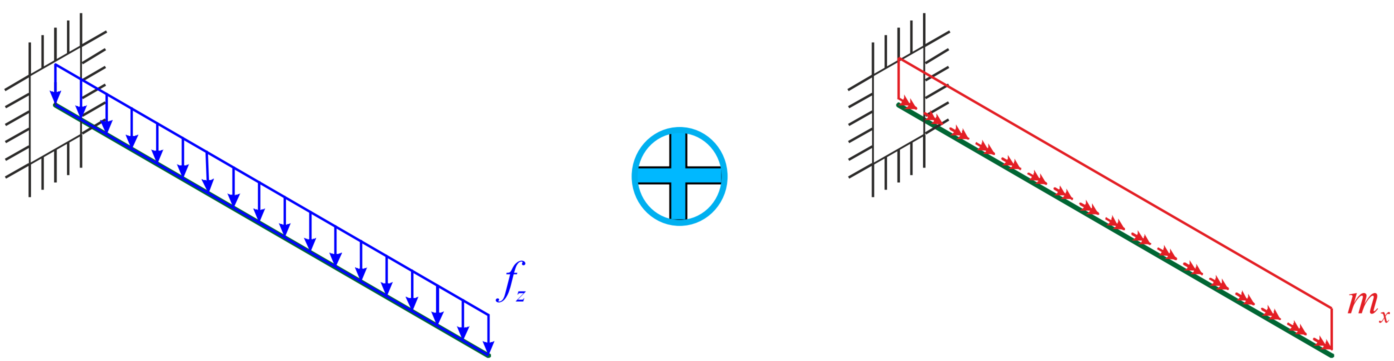


Beam segment ds

Transformation of dF_z

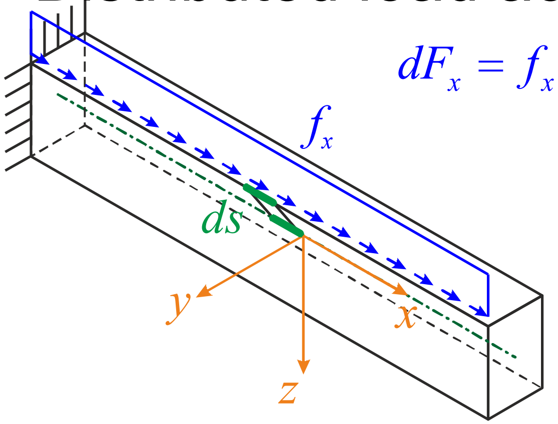
Additional distributed moment load m_x

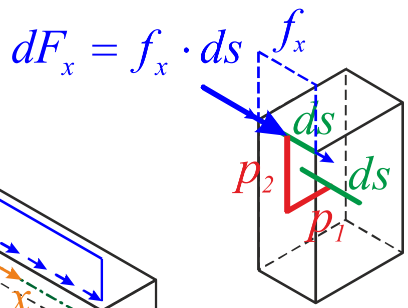
Model:



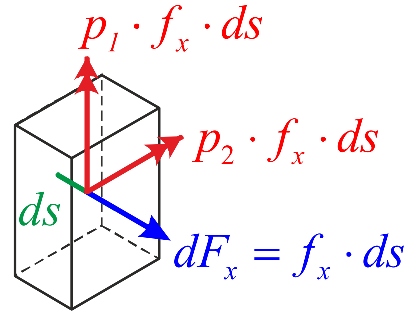
3D beams – load transformation

Distributed load acting on the line parallel to the centerline

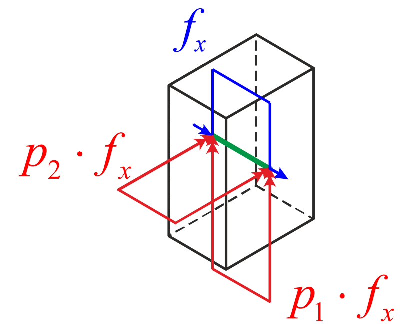




Beam segment *ds*

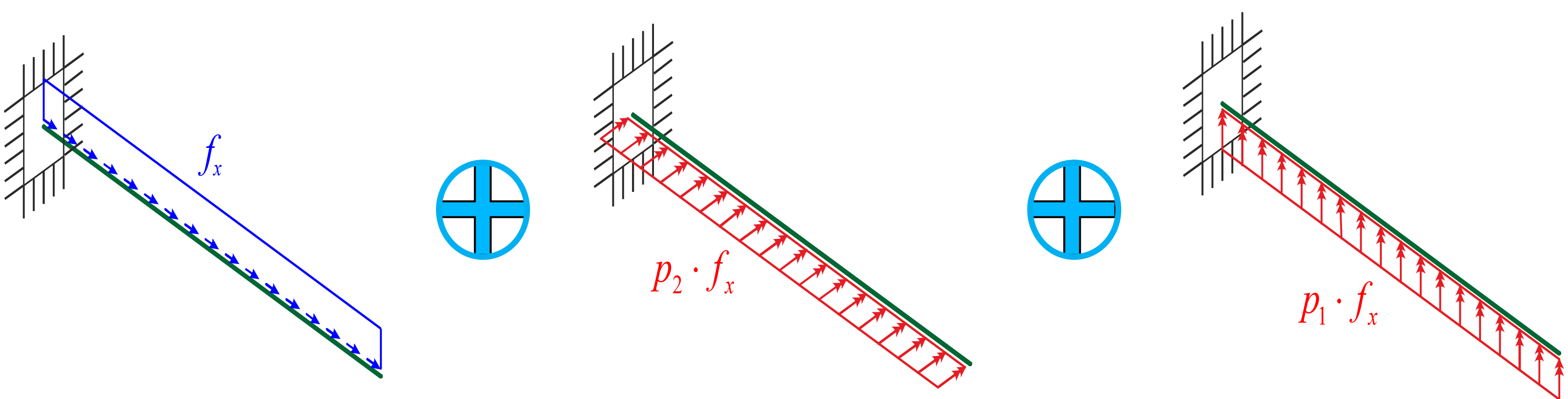


Transformation of dF_x



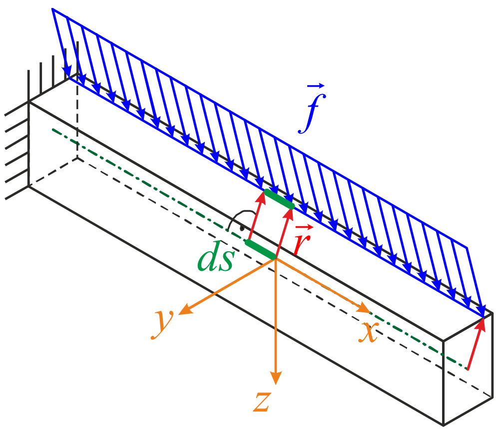
Additional distributed moment loads $p_2 \cdot f_x$ and $p_1 \cdot f_x$

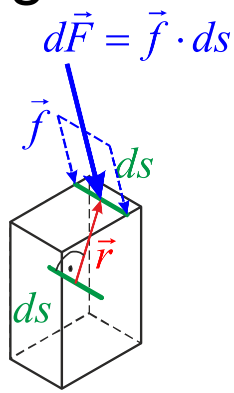
Model:



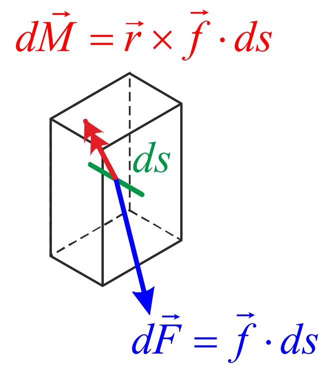
3D beams – load transformation

Distributed load acting on the line parallel to the centerline

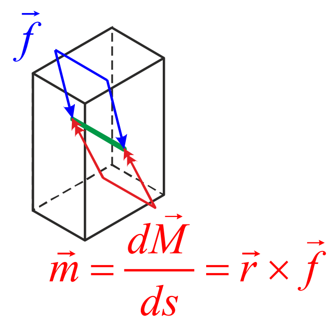




Beam segment *ds*

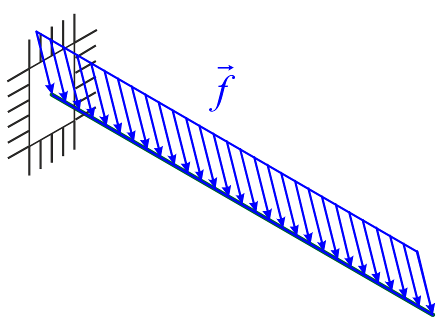


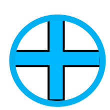
Transformation of $d\vec{F}$

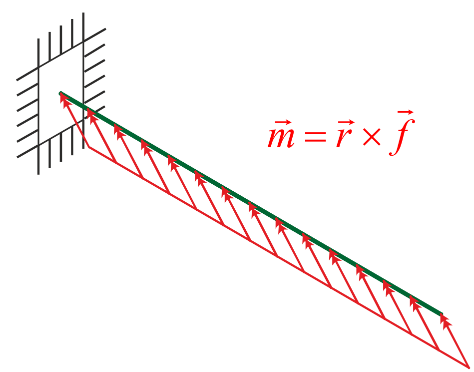


Additional distributed moment load $\vec{m}$

Model:

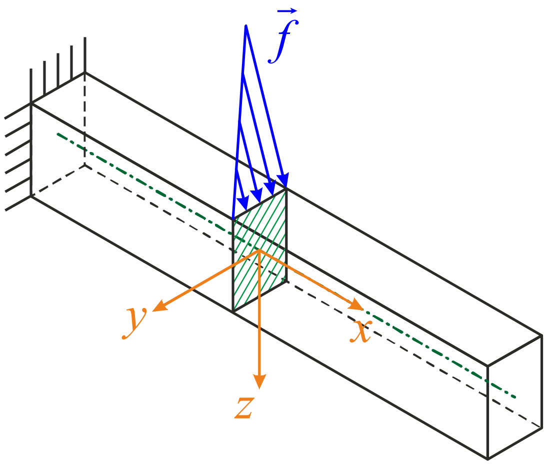


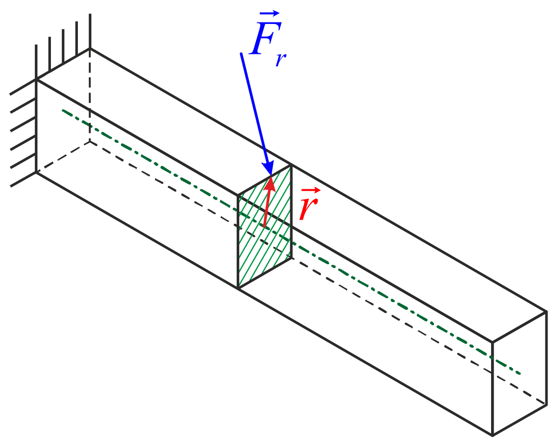


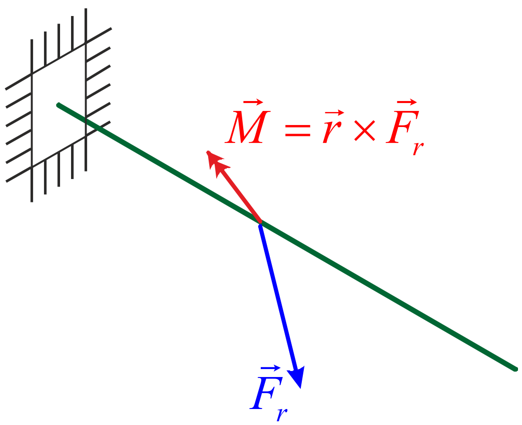


3D beams – load transformation

Distributed load acting on the line perpendicular to the centerline







... Load acting on the line which is in the plane of cross-section

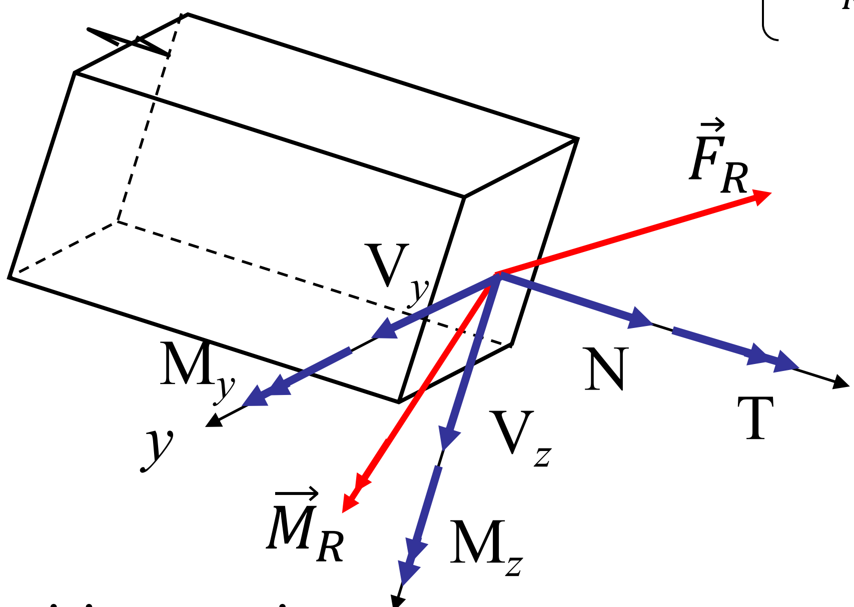
... Distributed load substituted by its resultant

... Transformation of this force to the center of gravity of the cross-section

3D beams – internal forces

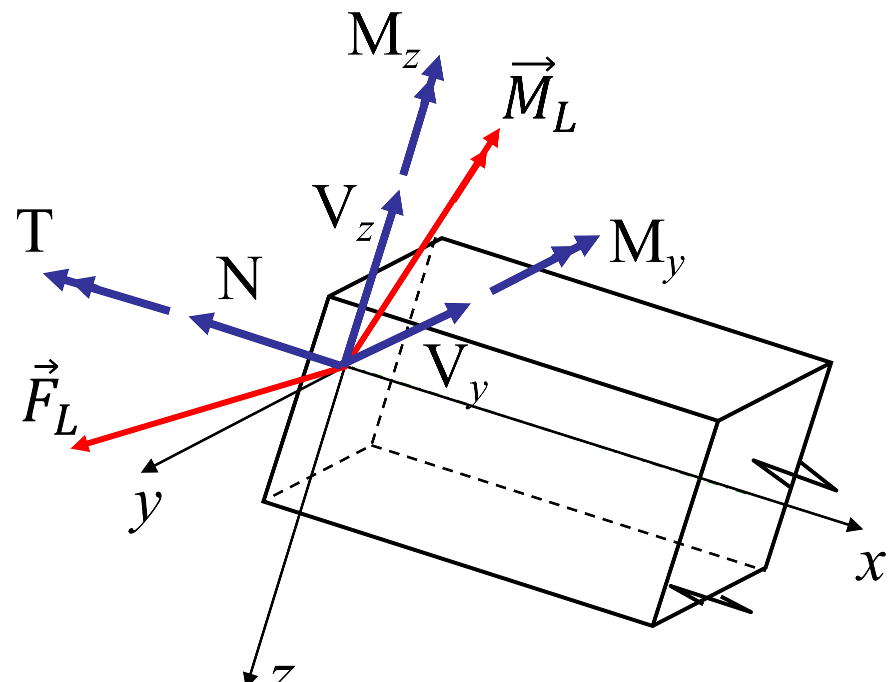
Orientation of positive internal forces

$$\vec{F}_R = -\vec{F}_L$$
$$\vec{M}_R = -\vec{M}_L$$

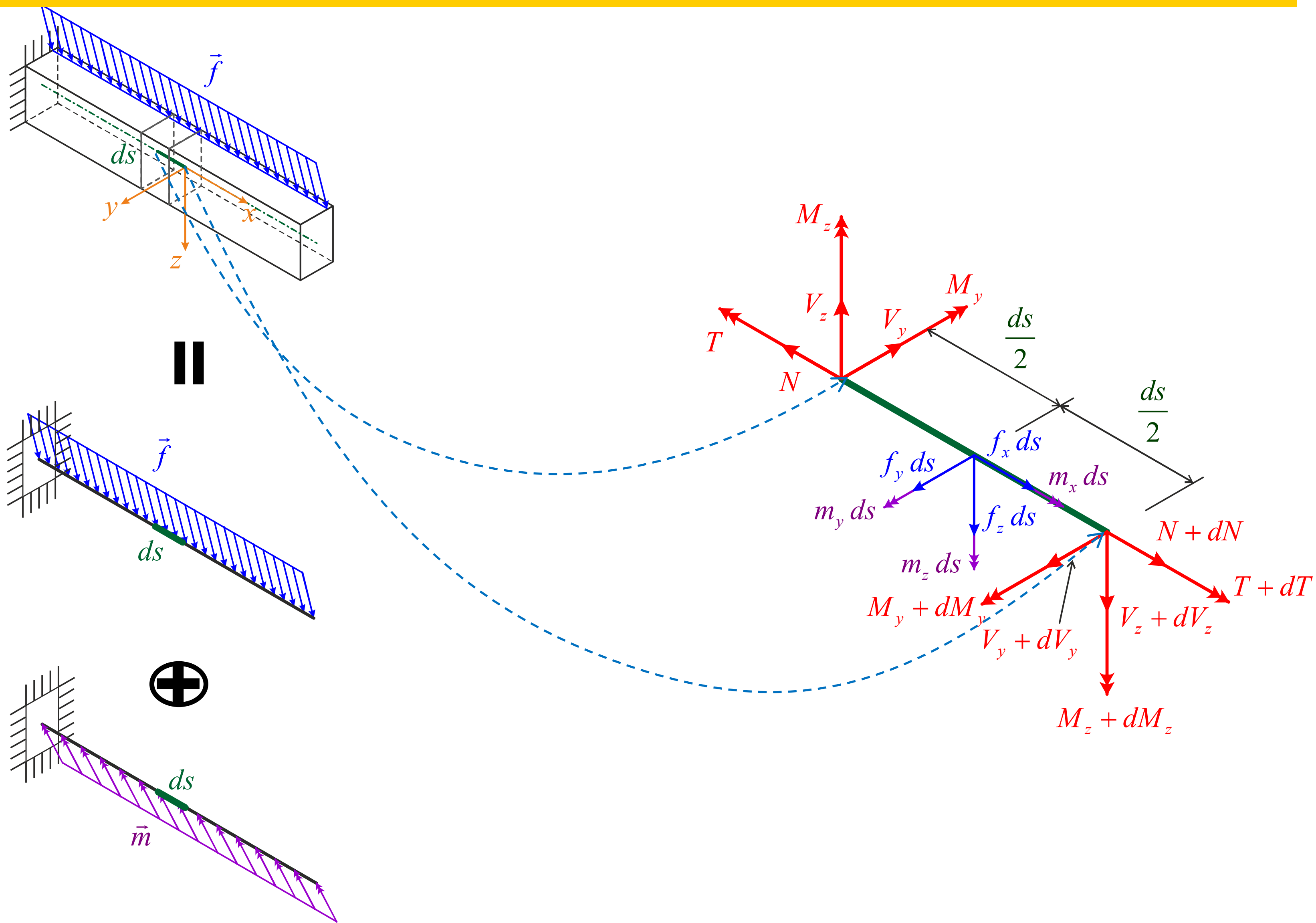


Positive section
(outer-pointing normal coincides with positive direction of x axis)

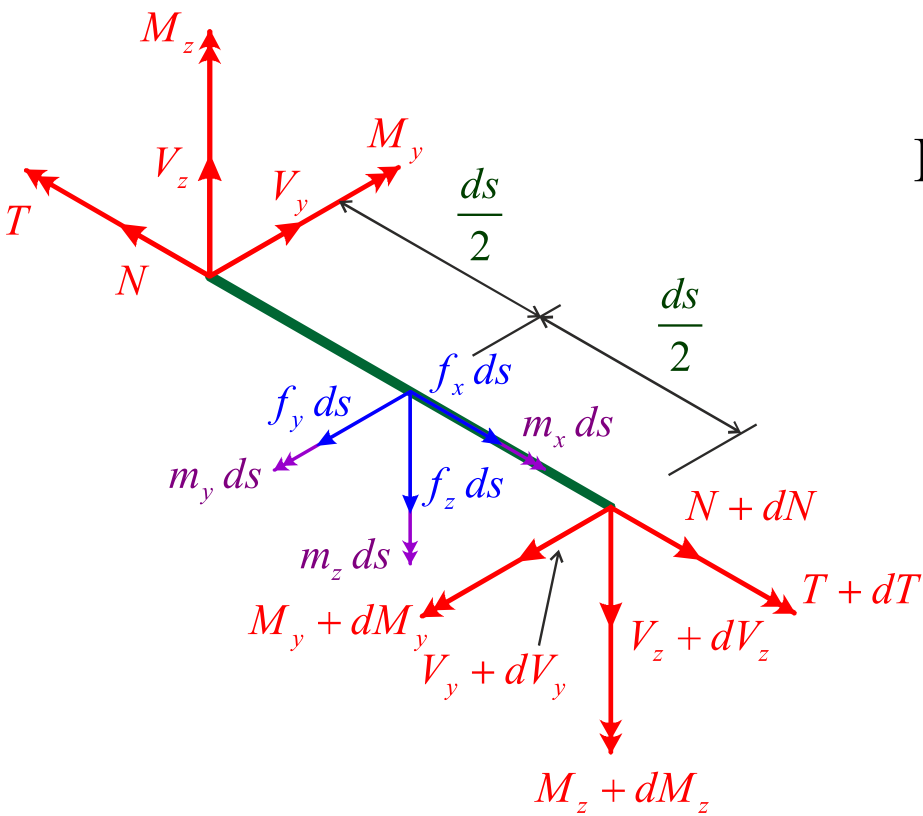
Negative section
(outer-pointing normal coincides with negative direction of x axis)



3D beams – generalization of relations between load and internal forces



3D beams – generalization of relations between load and internal forces



Equilibrium of beam element

$$x:\ -N + (N + dN) + f_x ds = 0$$

$$y:\ -V_y + (V_y + dV_y) + f_y ds = 0$$

$$z:\ -V_z + (V_z + dV_z) + f_z ds = 0$$

$$x:\ -T + (T + dT) + m_x ds = 0$$

$$y:\ -M_y + (M_y + dM_y) + m_y ds - V_z \frac{ds}{2} - (V_z + dV_z)\frac{ds}{2} = 0$$

$$z:\ -M_z + (M_z + dM_z) + m_z ds + V_y \frac{ds}{2} + (V_y + dV_y)\frac{ds}{2} = 0$$

3D beams – generalization of relations between load and internal forces

$$\searrow \; x: \; -N + (N + dN) + f_x ds = 0 \quad \Rightarrow \quad \frac{dN}{ds} = -f_x$$

$$\swarrow \; y: \; -V_y + (V_y + dV_y) + f_y ds = 0 \quad \Rightarrow \quad \frac{dV_y}{ds} = -f_y$$

$$\downarrow \; z: \; -V_z + (V_z + dV_z) + f_z ds = 0 \quad \Rightarrow \quad \frac{dV_z}{ds} = -f_z$$

$$\searrow\!\!\searrow \; x: \; -T + (T + dT) + m_x ds = 0 \quad \Rightarrow \quad \frac{dT}{ds} = -m_x$$

3D beams – generalization of relations between load and internal forces

$$y: \; -M_y + \left(M_y + dM_y\right) + m_y ds - V_z \frac{ds}{2} - \left(V_z + dV_z\right)\frac{ds}{2} = 0$$

$$dM_y + m_y ds - V_z ds - \underbrace{\cancel{dV_z \frac{ds}{2}}}_{\to 0} = 0$$

$$\boxed{\frac{dM_y}{ds} = V_z - m_y}$$

$$z: \; -M_z + \left(M_z + dM_z\right) + m_z ds + V_y \frac{ds}{2} + \left(V_y + dV_y\right)\frac{ds}{2} = 0$$

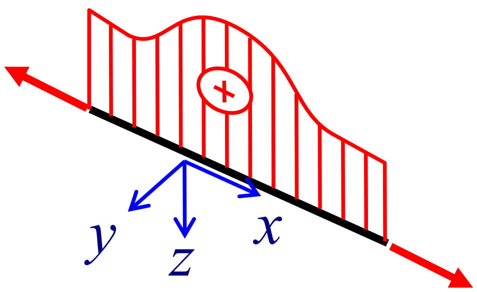
$$dM_z + m_z ds + V_y ds + \underbrace{\cancel{dV_y \frac{ds}{2}}}_{\to 0} = 0$$

$$\boxed{\frac{dM_z}{ds} = -V_y - m_z}$$

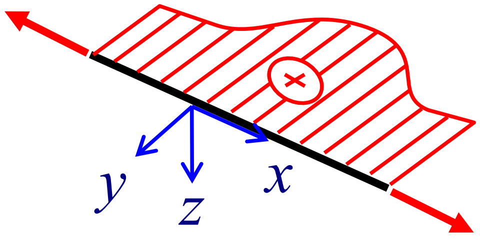
3D beams – plotting of internal forces

Normal force

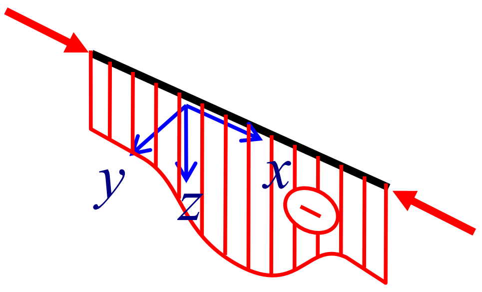
N > 0



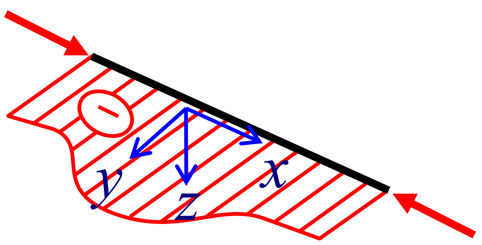
or



N < 0



or

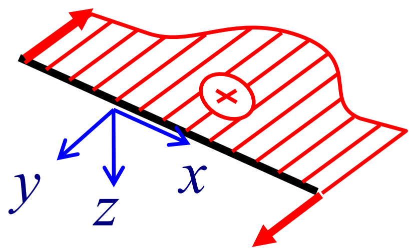


- the orientation and the plane of the sketch is not prescribed

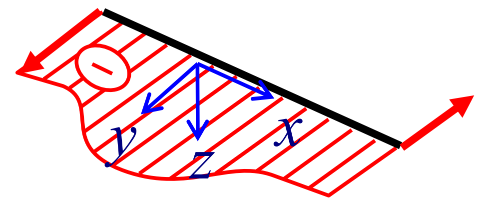
3D beams – plotting of internal forces

Shear forces

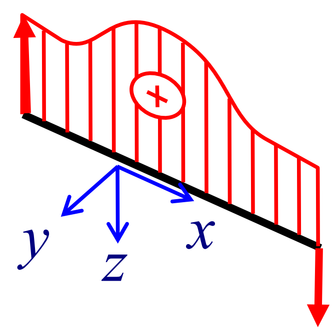
$V_y > 0$



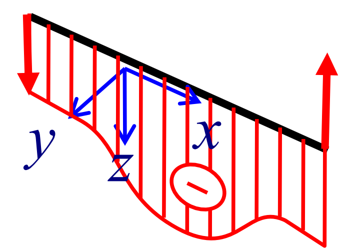
$V_y < 0$



$V_z > 0$



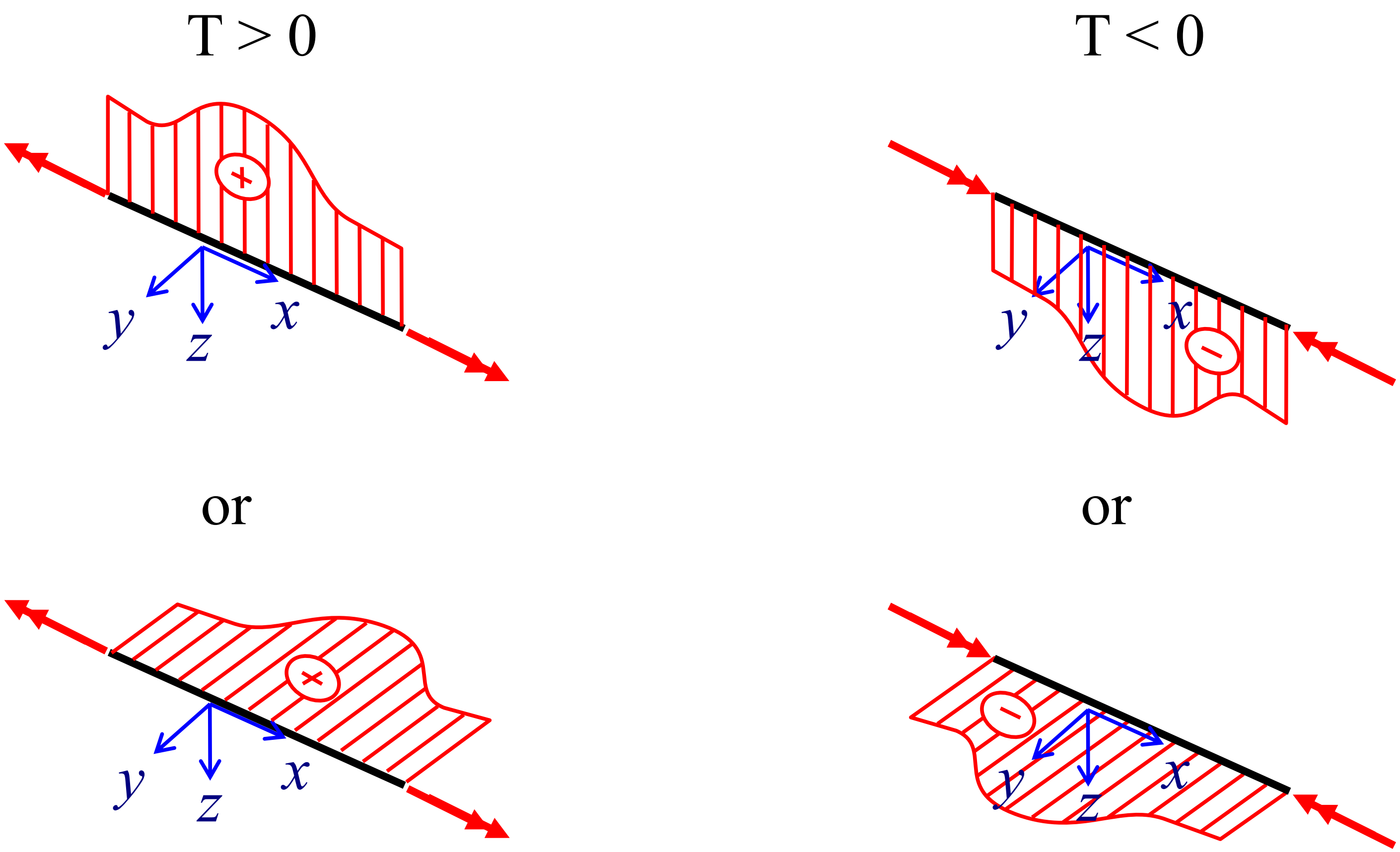
$V_z < 0$



- The orientation is not prescribed
- The plane is prescribed (plane in which the forces act)

3D beams – plotting of internal forces

Torque

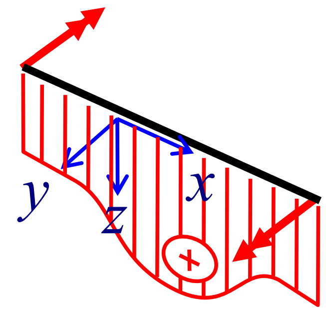


- the orientation and the plane of the sketch is not prescribed

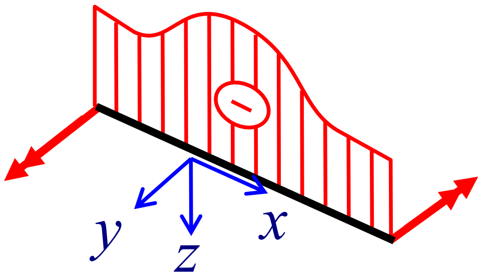
3D beams – plotting of internal forces

Bending moments

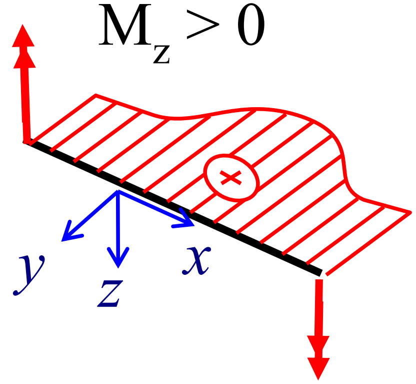
$M_y > 0$



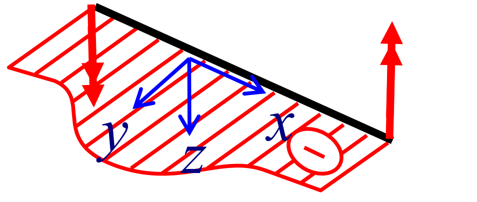
$M_y < 0$



$M_z > 0$



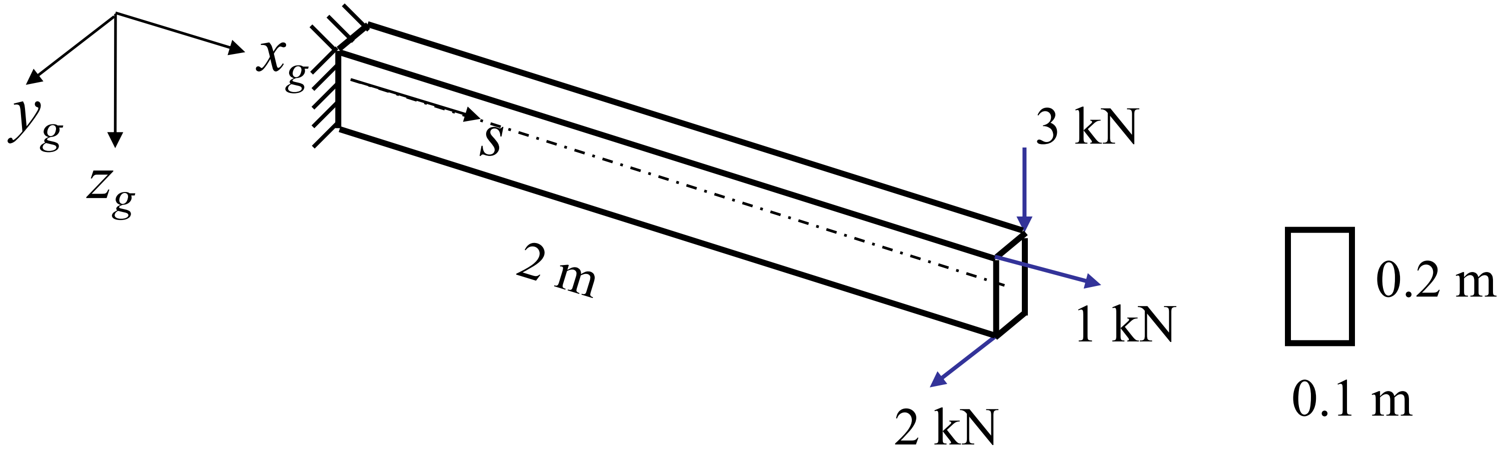
$M_z < 0$



- the orientation and the plane of the sketch is prescribed (on the side where the tension is)

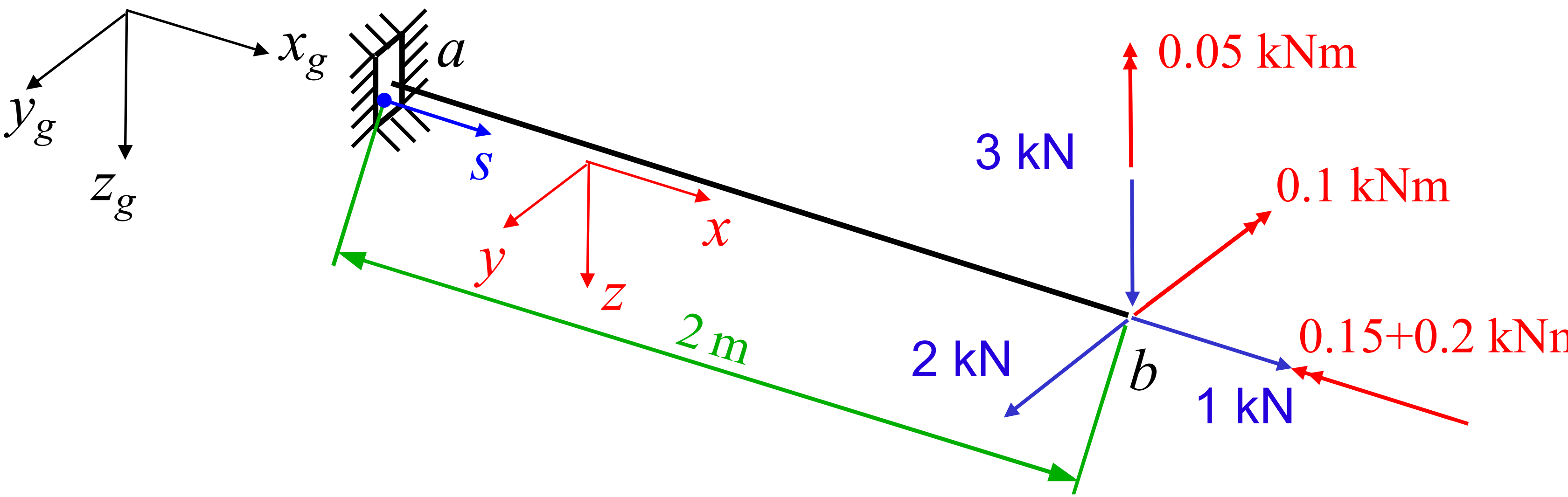
3D beams – example

Determine and plot internal forces along the beam



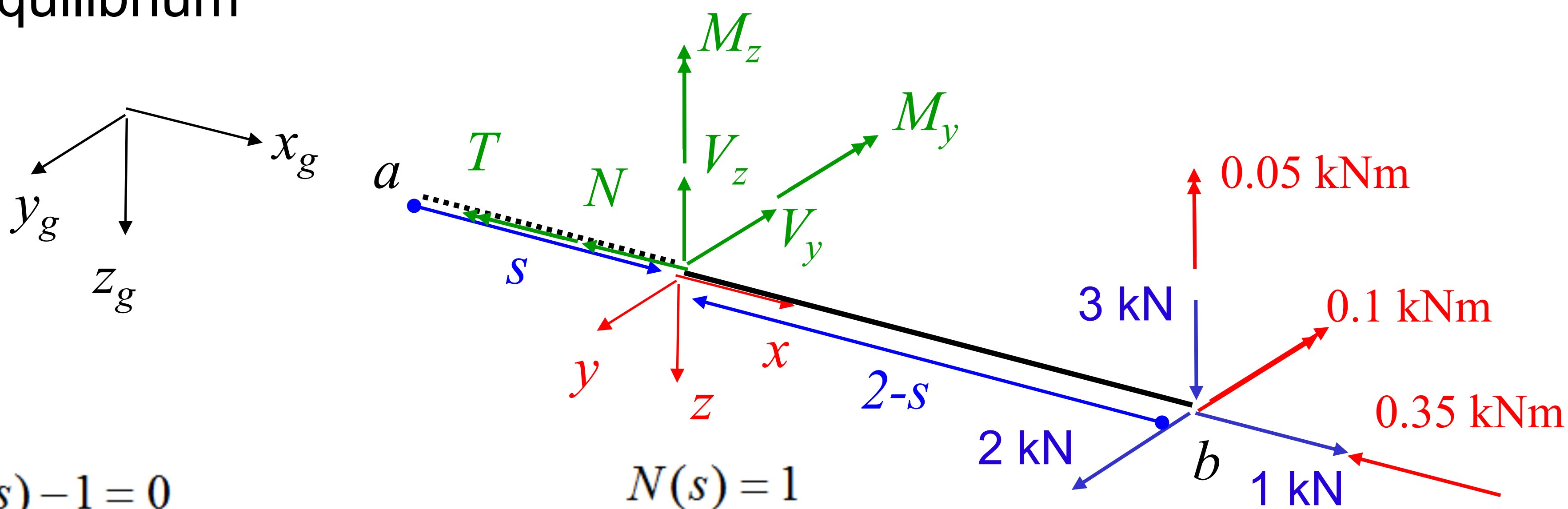
3D beams – example

Model (transformation of the load):



3D beams – example

Equilibrium



$$N(s)-1=0 \qquad N(s)=1$$

$$V_y(s)-2=0 \qquad V_y(s)=2$$

$$V_z(s)-3=0 \qquad V_z(s)=3$$

$$T(s)+0.35=0 \quad \Rightarrow \quad T(s)=-0.35$$

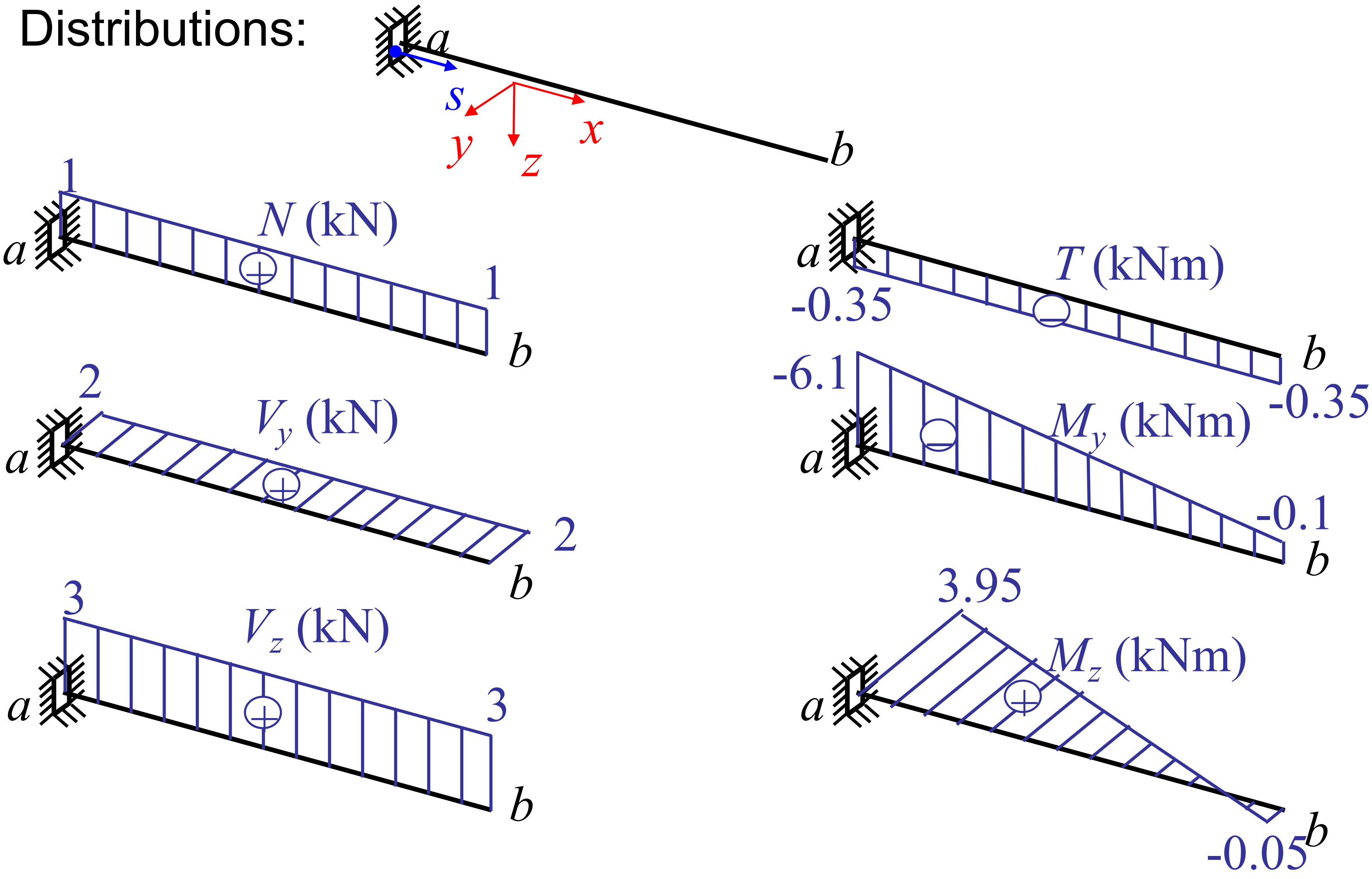
$$M_y(s)+0.1+3(2-s)=0 \qquad M_y(s)=-6.1+3s$$

$$M_z(s)+0.05-2(2-s)=0 \qquad M_z(s)=3.95-2s$$

[kN, kNm]

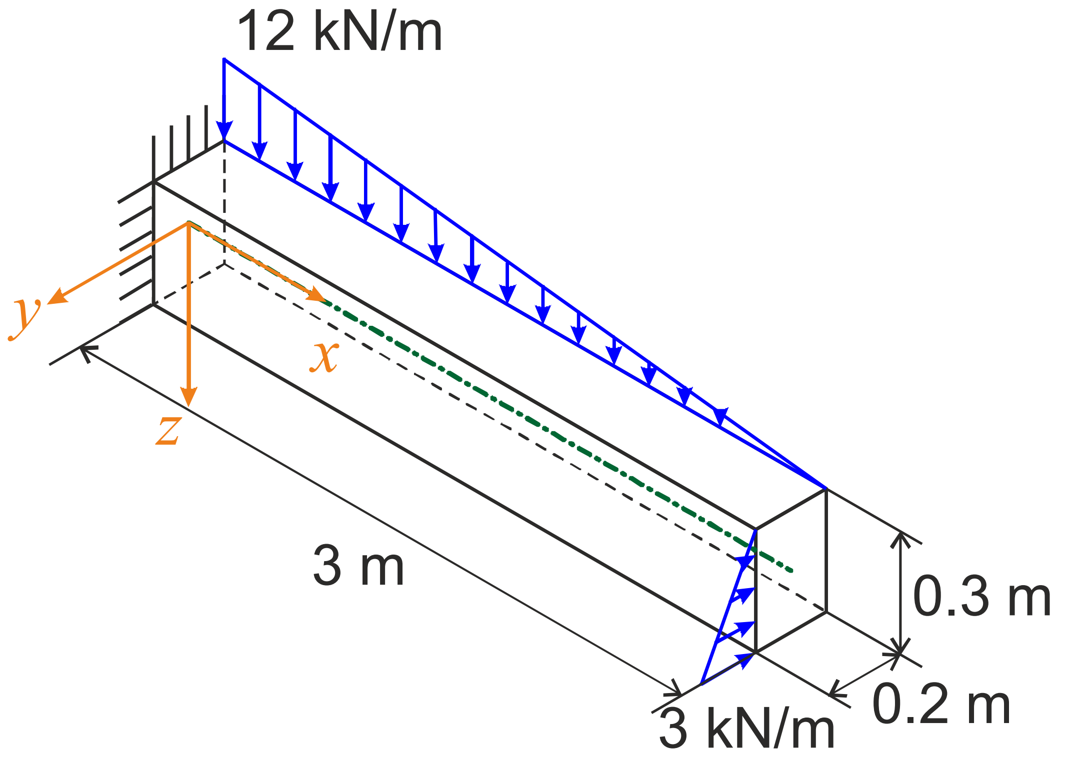
3D beams – example

Distributions:



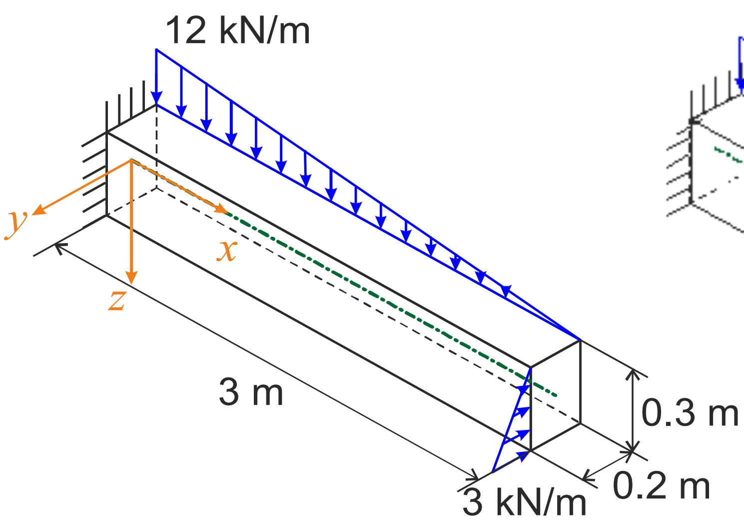
3D beams – example

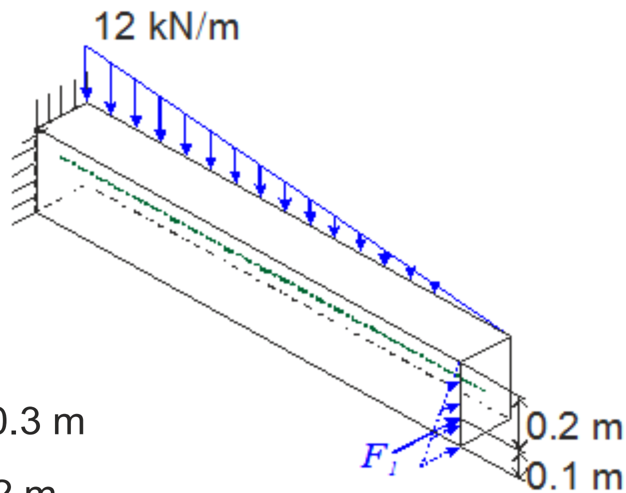
Determine and plot internal forces along the beam

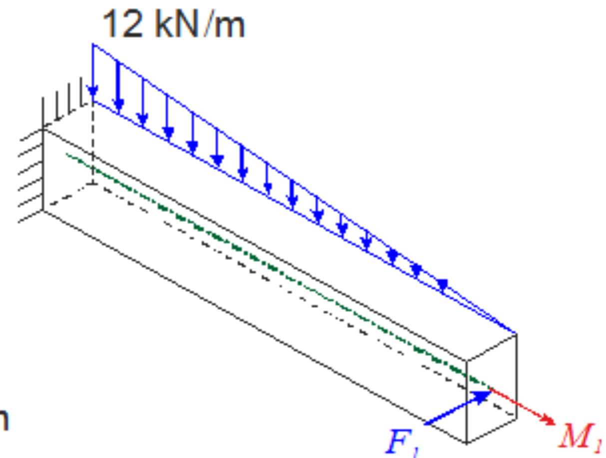


3D beams – example

Model (transformation of load):

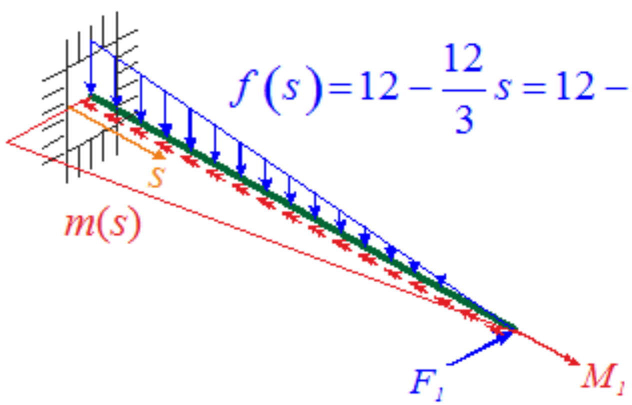






$$F_1 = \frac{3 \cdot 0.3}{2} = 0.45\,kN$$

$$M_1 = 0.45 \cdot 0.05 = 0.0225\,kNm$$

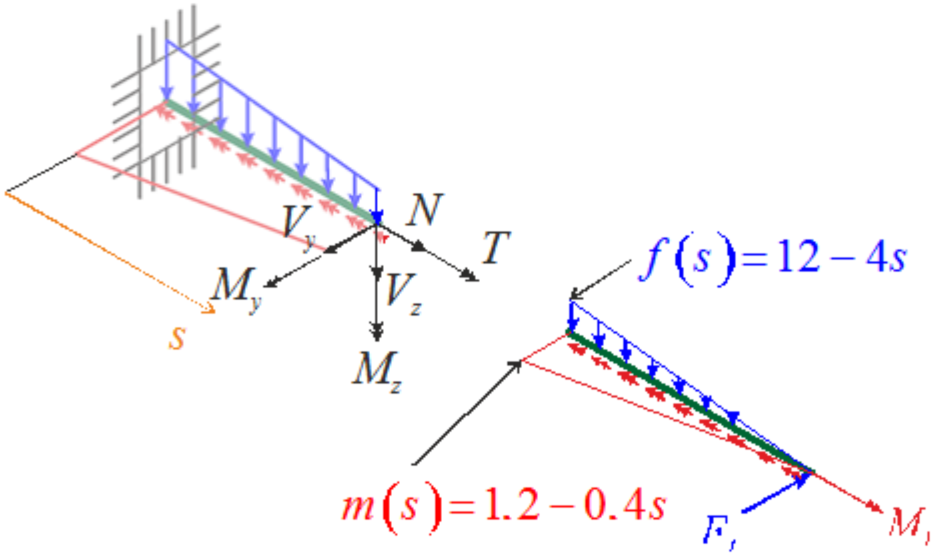


$$f(s) = 12 - \frac{12}{3}s = 12 - 4s \quad [kN/m]$$

$$m(s) = f \cdot 0.1 = 1.2 - 0.4s \quad [kNm/m]$$

3D beams – example

Equivalency



$$N(s)=0 \quad [kN]$$

$$V_y(s)=-F_1=-0.45 \quad [kN]$$

$$V_z(s)=\frac{f(s)\cdot(l-s)}{2}=\frac{(12-4s)(3-s)}{2}=2s^2-12s+18 \quad [kN]$$

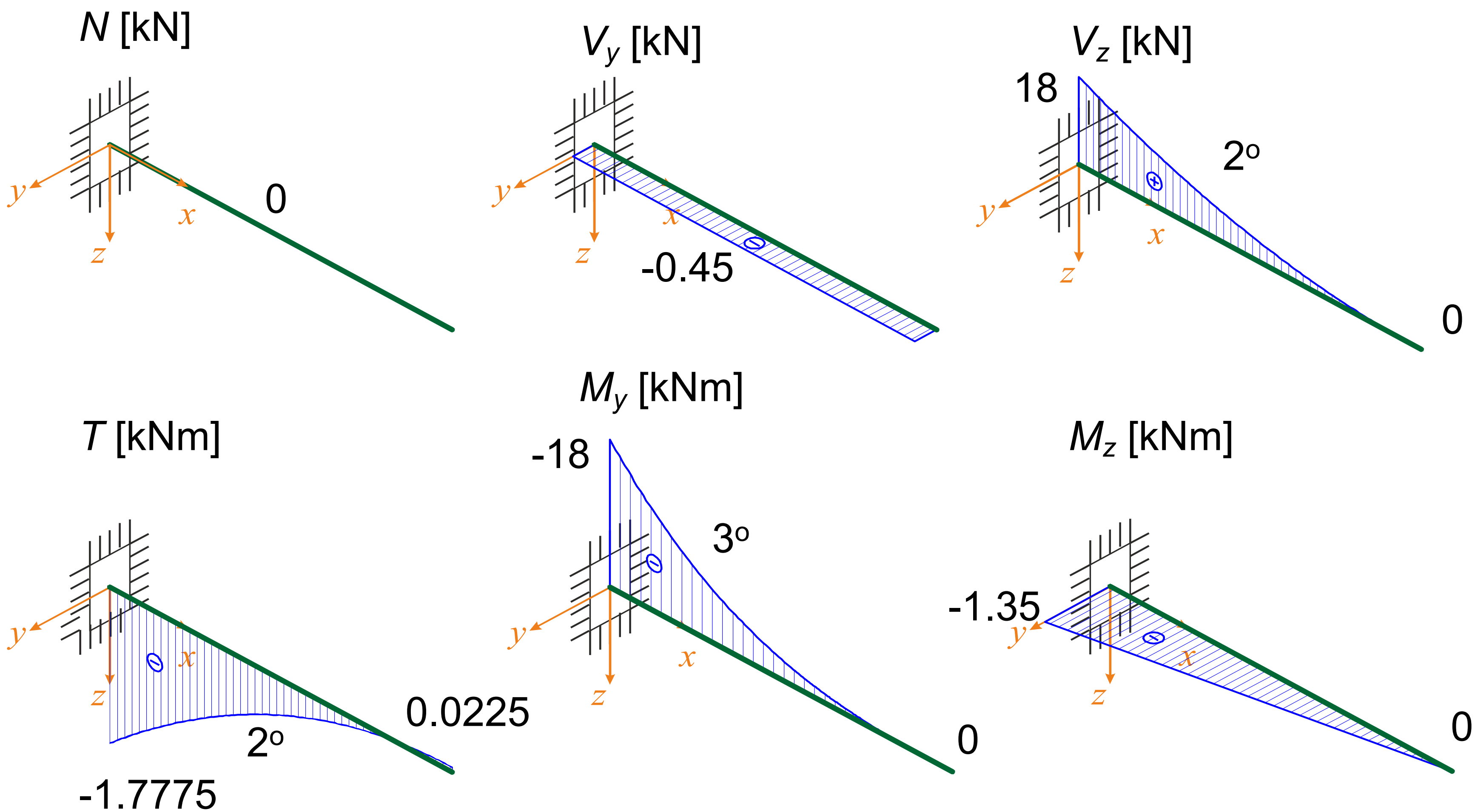
$$T(s)=-\frac{m(s)\cdot(l-s)}{2}+M_1=-\frac{(1.2-0.4s)(3-s)}{2}+0.0225=-0.2s^2+1.2s-1.7775 \quad [kNm]$$

$$M_y(s)=-\frac{f(s)\cdot(l-s)}{2}\cdot\frac{(l-s)}{3}=-\frac{(12-4s)(3-s)^2}{6}=\frac{2s^3-18s^2+54s-54}{3} \quad [kNm]$$

$$M_z(s)=-F_1\cdot(l-s)=-0.45\cdot(3-s)=-1.35+0.45s \quad [kNm]$$

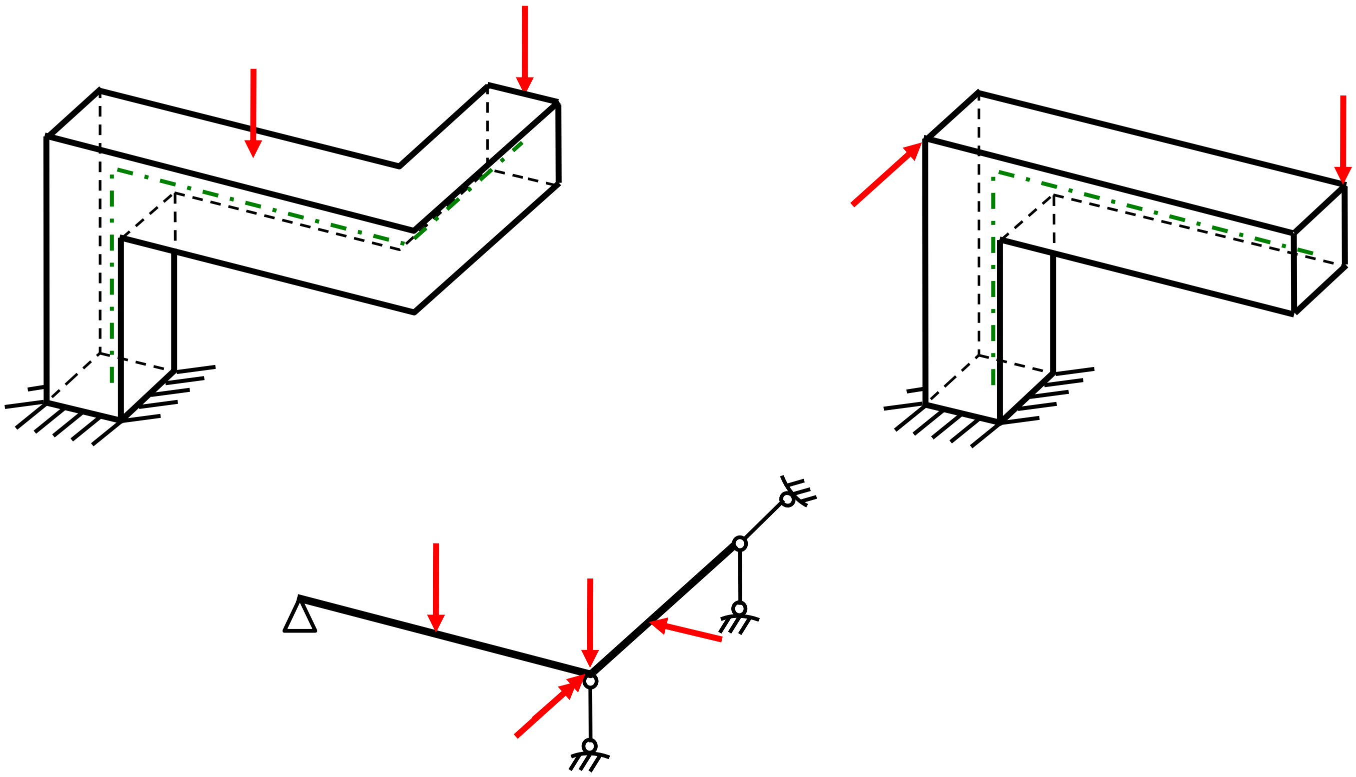
3D beams – example

Distributions:

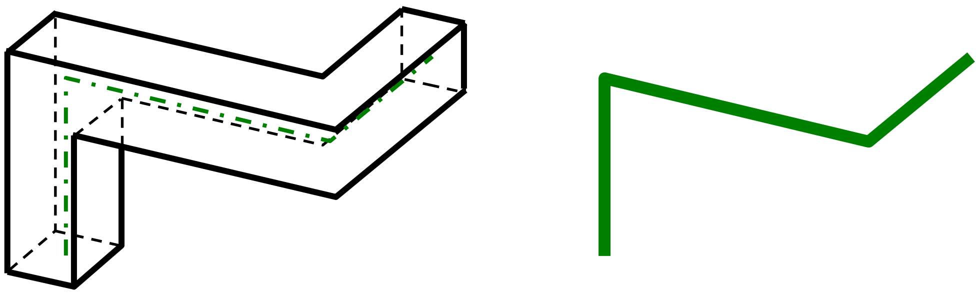


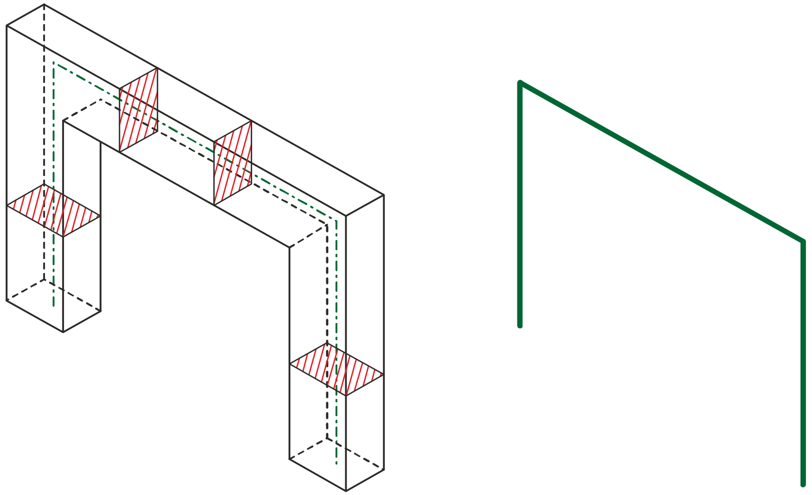
3D cranked beams

- centerline: polyline line (2D or 3D)
- load and reactions: force vectors do not act in the plane
  of the centreline
- Eg.:



3D cranked beams - model

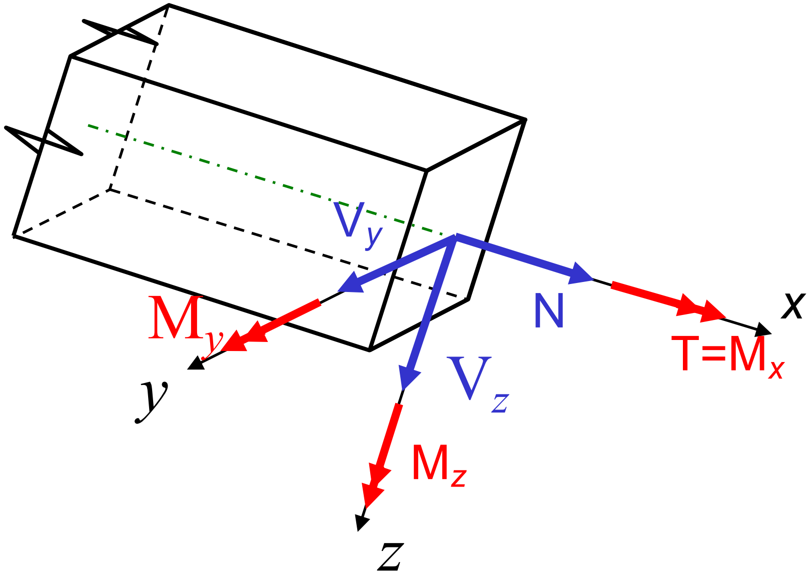




Transformation of load

- same as for straight beams

3D cranked beams – internal forces

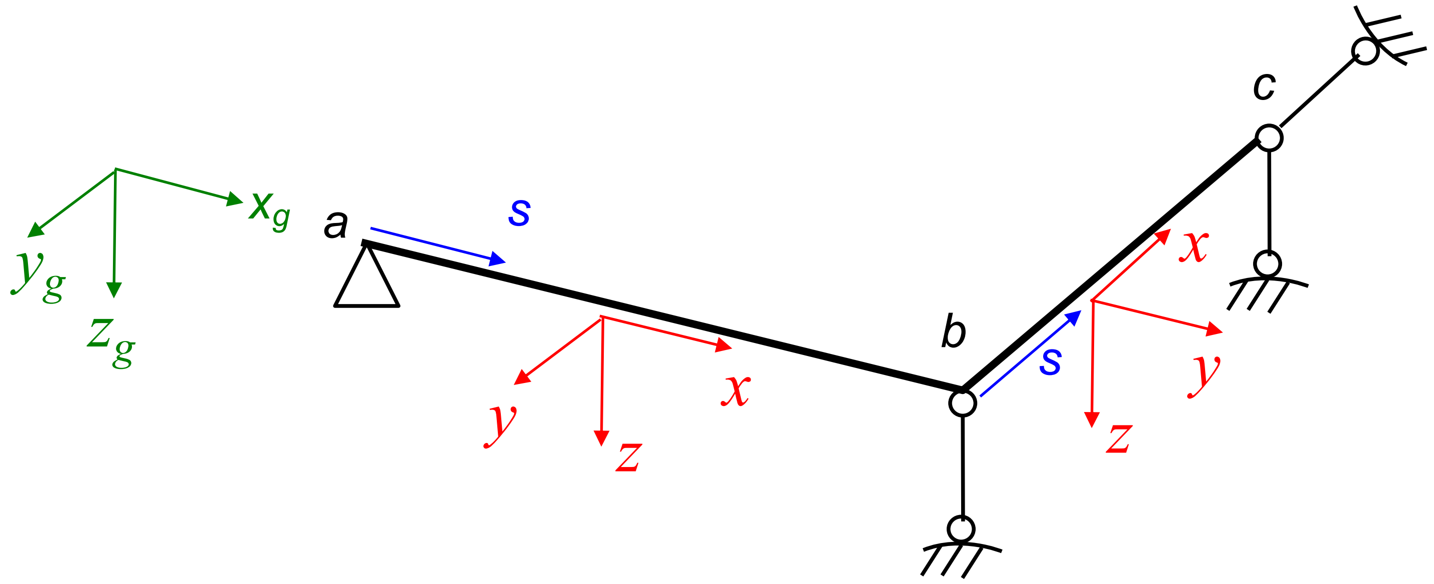


Division of beam into intervals:

- change of load
- point force or moment
- support, connection, joint
- end of beam

3D cranked beams

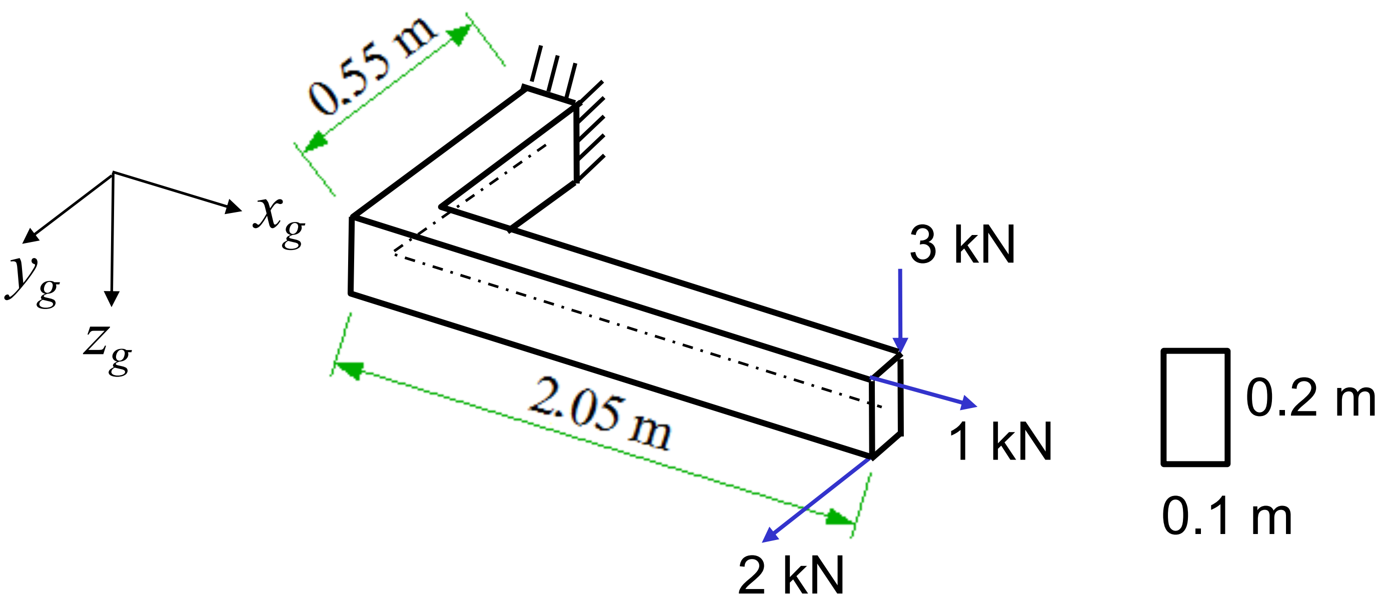
- Coordinate systems
  - global x_g-y_g-z_g
  - local x-y-z (orientation and direction of internal forces)
  - local s (internal forces as a function of cross-section position)



- Plotting of distributions of internal forces: same as for straight beams

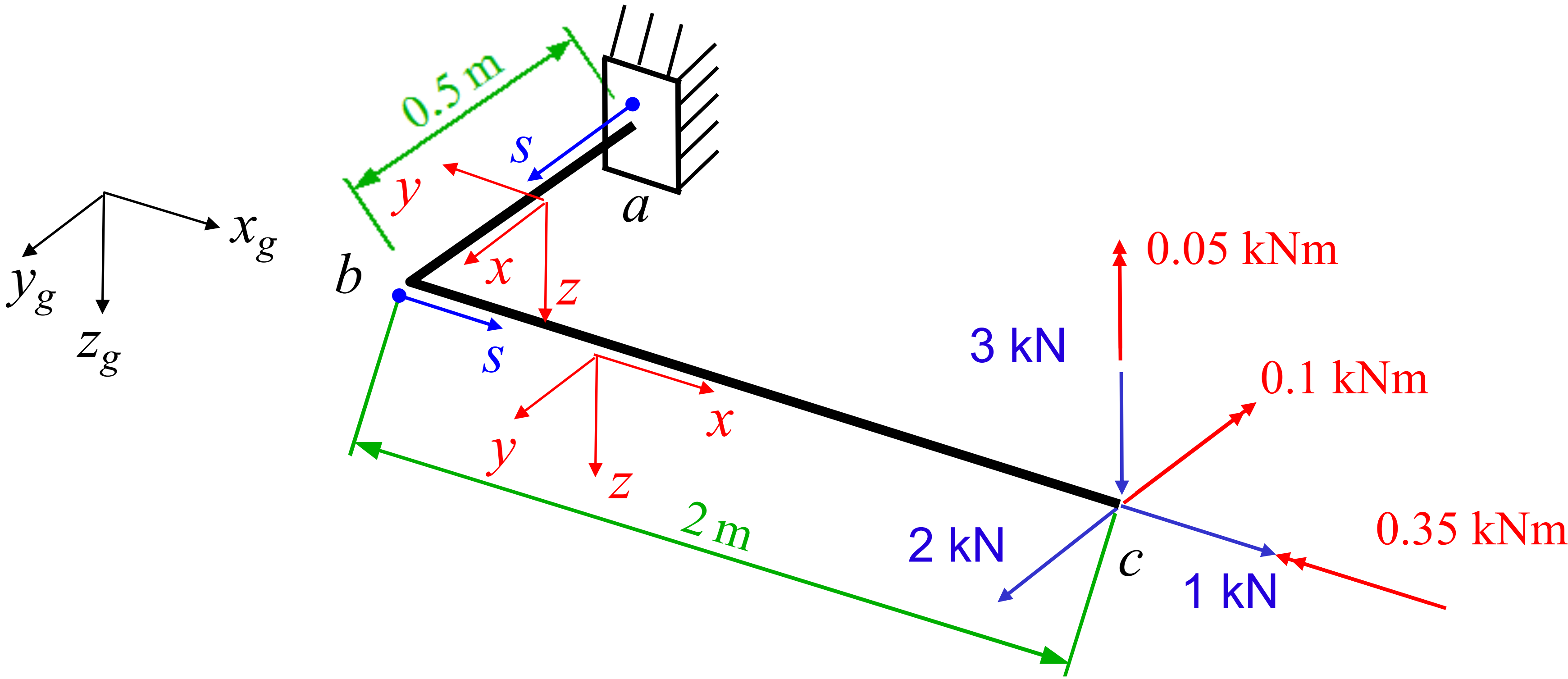
3D cranked beams – example

Determine and plot internal forces along the beam



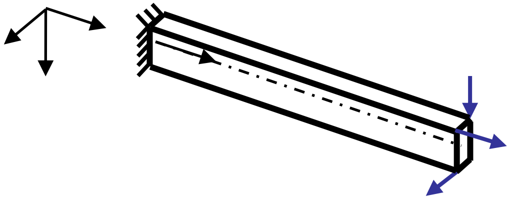
3D cranked beams – example

Model:



3D cranked beams – example

Interval (*b*,*c*) ... see example in previous section



$$N(s) = 1 \quad \mathrm{kN}$$

$$V_y(s) = 2 \quad \mathrm{kN}$$

$$V_z(s) = 3 \quad \mathrm{kN}$$

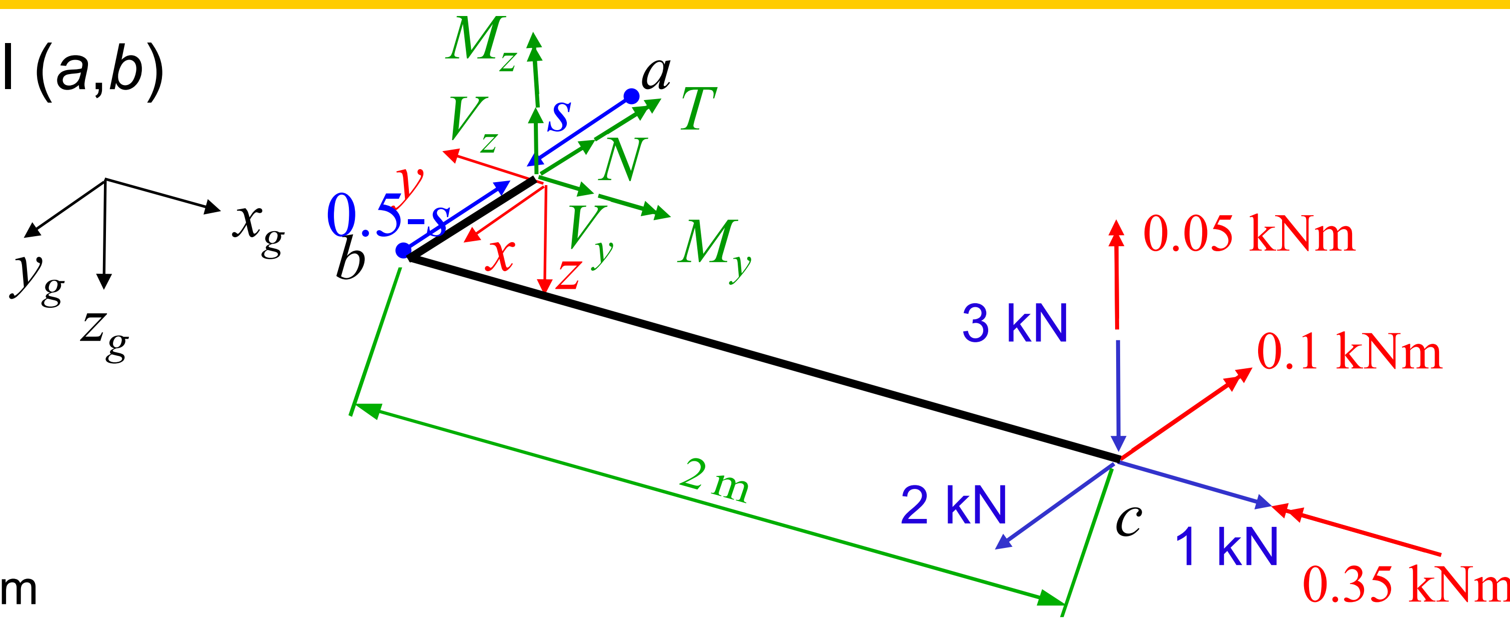
$$T(s) = -0.35 \quad \mathrm{kNm}$$

$$M_y(s) = -6.1 + 3s \quad \mathrm{kNm}$$

$$M_z(s) = 3.95 - 2s \quad \mathrm{kNm}$$

3D cranked beams – example

Interval (a,b)



Equilibrium

$$N(s)-2=0 \qquad N(s)=2$$
$$V_y(s)+1=0 \qquad V_y(s)=-1$$
$$V_z(s)-3=0 \quad \Rightarrow \quad V_z(s)=3$$
$$T(s)+3\cdot 2+0.1=0 \qquad T(s)=-6.1$$
$$M_y(s)-0.35+3(0.5-s)=0 \qquad M_y(s)=-1.15+3s$$
$$M_z(s)+0.05-2\cdot 2+1(0.5-s)=0 \qquad M_z(s)=3.45+s$$

[kN, kNm]

The End