Structural mechanics 2

Summer semester 2023/24

Structural mechanics 2

Lecture no. 8, April 16, 2024

1) Moments of inertia
2) Mixed moment of inertia
3) Radius of gyration, ellipse of inertia

Review – lecture No. 7

Models of structural systems

Model of real structure – **numerical model**

- Simplification, idealization of reality

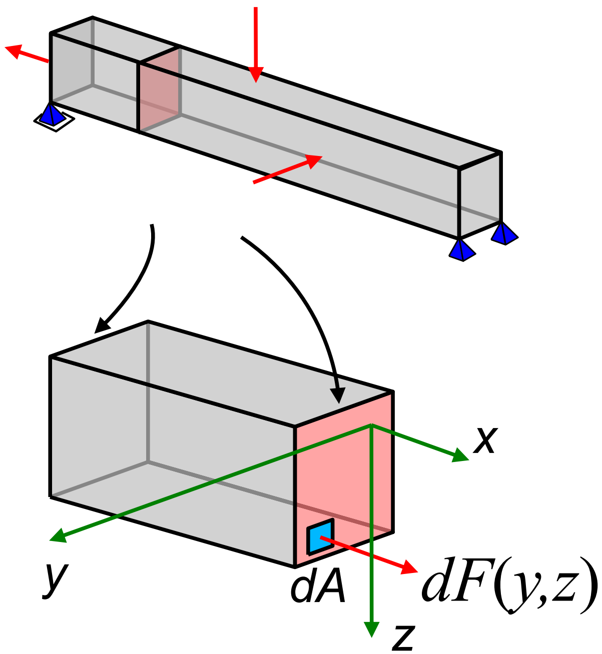
a) Geometry of the structure => 3D → 2D → 1D, division of structure

b) Type of joints, connections, etc. => stiffness → 0, → ∞

c) Load => mechanical and non-mechanical load

! ! The simplification of the structure cannot omit any specific detail and must appropriately represent the structure. ! !

Normal stress – force equivalency



The load carried by the cross-section is described by internal forces N, V_y, V_z, T, M_y, M_z

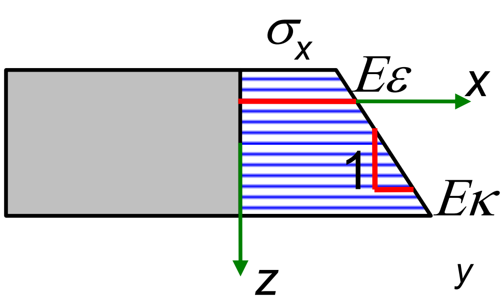
To describe the local force contribution in each point of cross-section the **normal stress** is defined:

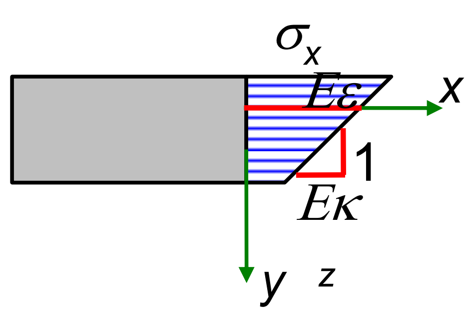
$$\sigma_x(y,z) = \frac{dF(y,z)}{dA}$$

Theory of elasticity:
Linear distribution of normal stress over the cross-section:

$$\sigma_x(y,z) = E\left(\varepsilon - \kappa_z y + \kappa_y z\right) \quad (1)$$

beam curvature
strain
Young's modulus
(material stiffness)





Normal stress – force equivalency

To be able to calculate the normal stress, we need to determine:

1. Centroid of cross-section (using $S_y = S_z = 0$)

2. Principal axes (using $D_{yz} = 0$)

3. Moments of inertia w.r.t. principal centroidal axes

4. Internal forces w.r.t. principal centroidal axes

Centroid – area

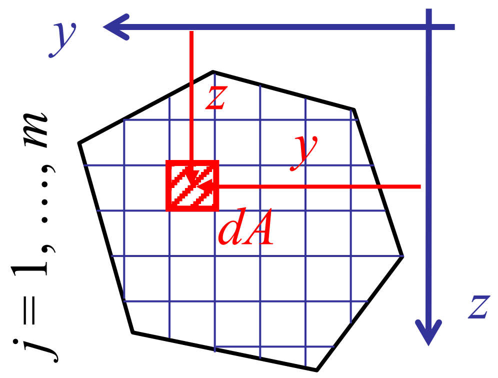
- Limit transition: $m \to \infty$, $n \to \infty$ => fine division

$$A = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta A = \iint_A dA$$

$$S_z = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij} \Delta A = \iint_A y \, dA$$

$$S_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} z_{ij} \Delta A = \iint_A z \, dA$$

(double integral)



$i = 1, \ldots, m$

- Centroid coordinates (analogous to system of material points):

$$y_c = \frac{\iint_A y \cdot dA}{A} = \frac{S_z}{A}$$

$$z_c = \frac{\iint_A z \cdot dA}{A} = \frac{S_y}{A}$$

Lecture No. 8

Moment of inertia
Mixed moment of inertia

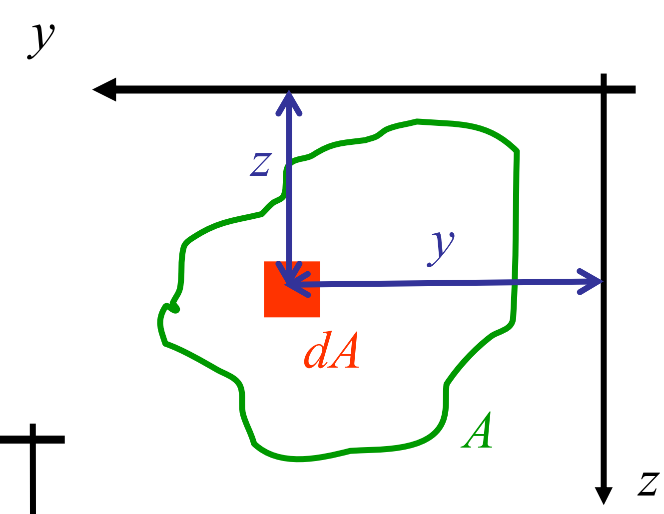
First moment of area

- y axis $S_y = \int_A z dA$ - z axis $S_z = \int_A y dA$ [m³]

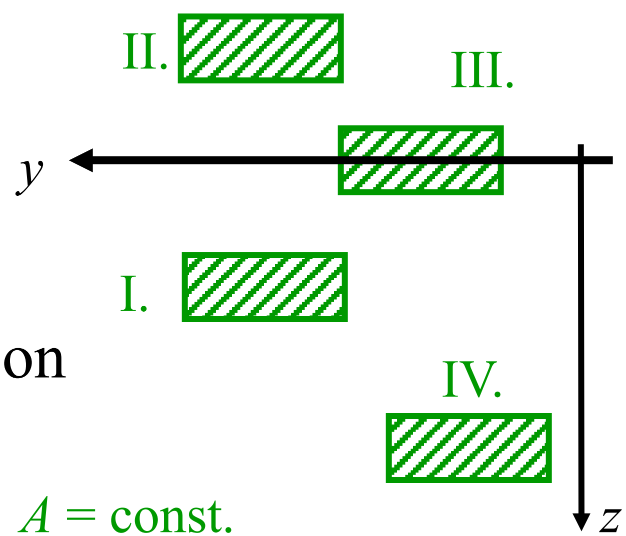
Moment of inertia

- y axis $I_y = \int_A z^2 dA$ [m⁴]

-z axis $I_z = \int_A y^2 dA$ [m⁴]



E.g.:



1) $I_y, I_z > 0$

2) I_y, I_z is dependent on the distance of the area from axis

A = const.

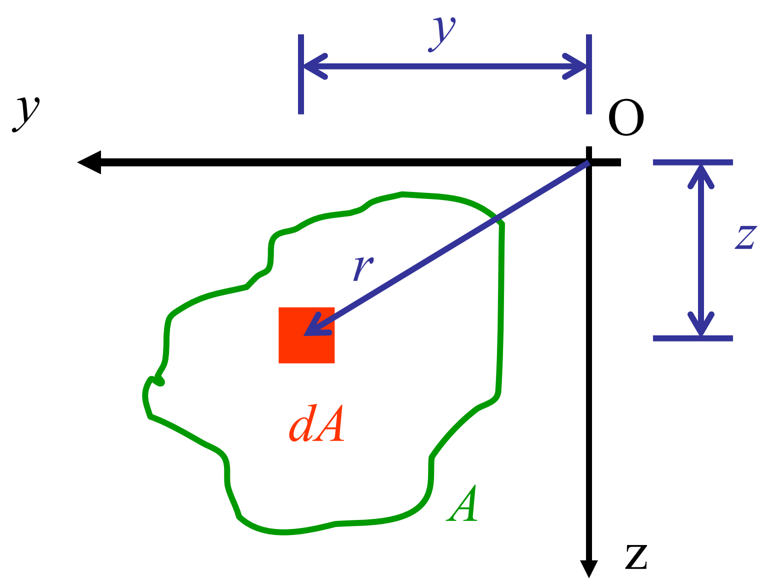
$$I_y^{I} = I_y^{II}$$

$$I_y^{III} < I_y^{I} = I_y^{II} < I_y^{IV}$$

Moment of inertia
Mixed moment of inertia

Polar moment of inertia

-w.r.t. origin "O"



$$I_0 = \int_A r^2 dA = \int_A \left(y^2 + z^2 \right) dA$$

[m⁴]

Note: $I_0 = I_z + I_y$

Moment of inertia
Mixed moment of inertia

Mixed moment of inertia

- y, z axes

$$D_{yz} = \int_A yz\,dA \qquad [m^4]$$

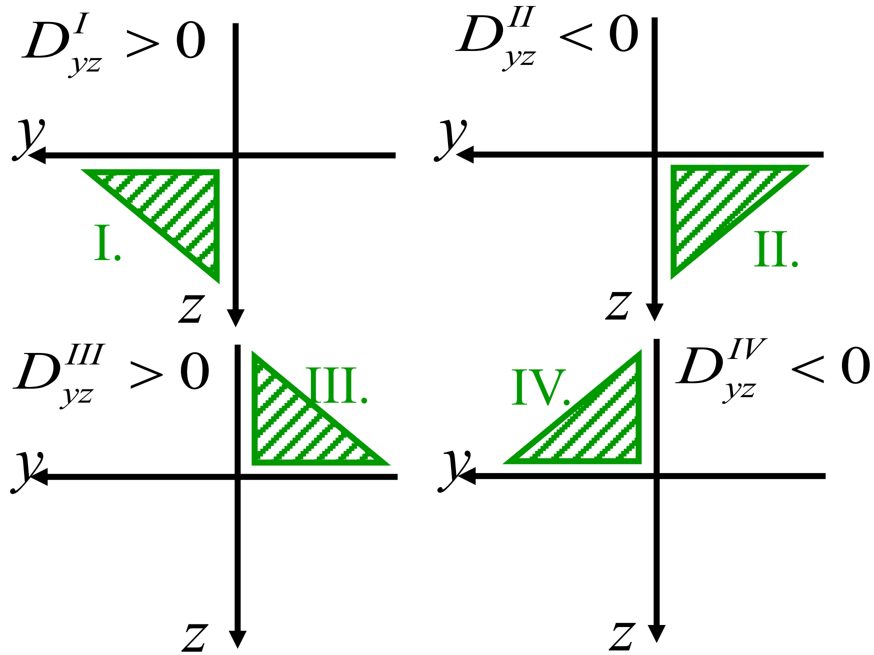
Note:

1) $D_{yz} \lesseqgtr 0$

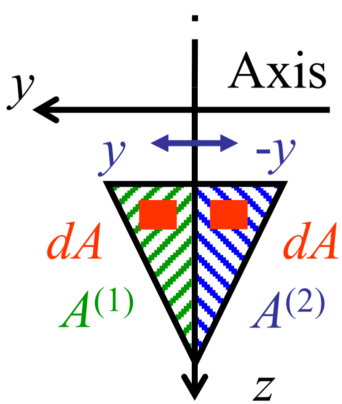
2) symmetrically placed area

$$D_{yz}^{I} = -D_{yz}^{II} = D_{yz}^{III} = -D_{yz}^{IV}$$

$$D_{yz}^{I} > 0$$



3) y or z ... axis of symmetry => $D_{yz} = 0$



Axis of symmetry = z

$$\int_A yz\,dA = \int_{A^{(1)}} yz\,dA + \int_{A^{(2)}} yz\,dA$$

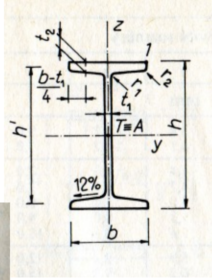
$$\int_{A^{(1)}} yz\,dA = -\int_{A^{(2)}} yz\,dA \Rightarrow D_{yz} = 0$$

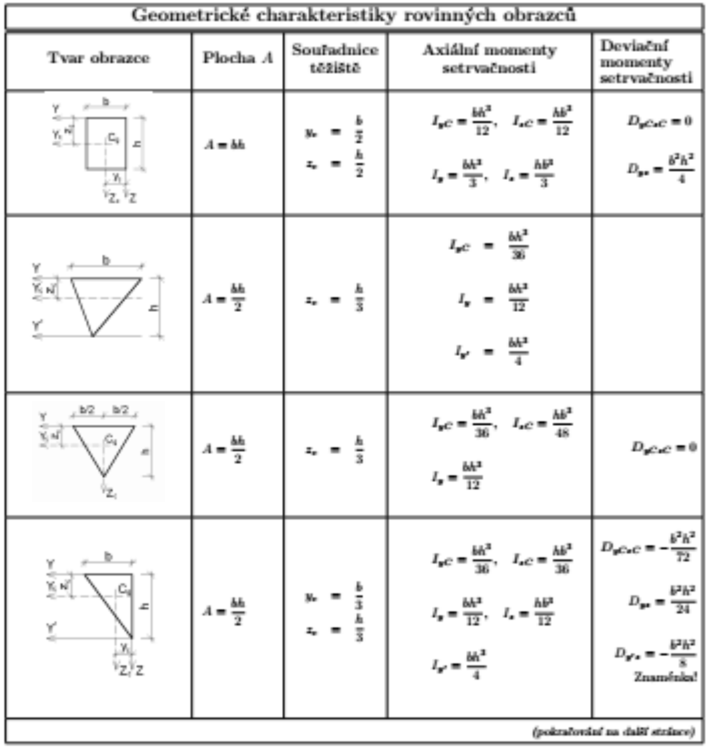
Moment of inertia
Mixed moment of inertia

Calculation:

1. Integration

2. Combination of elementary geometric shapes



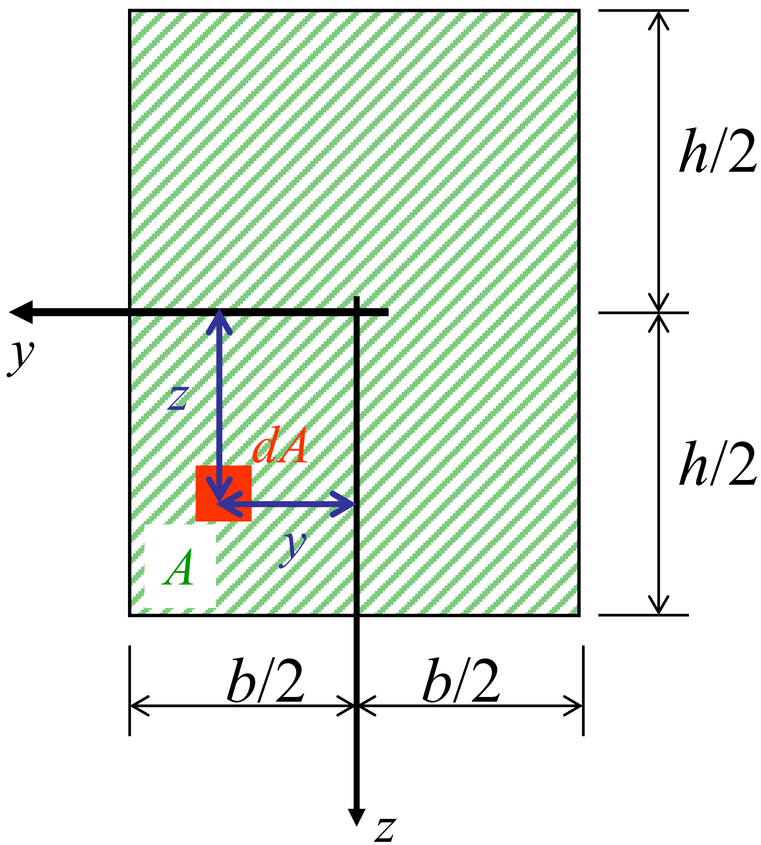


Statické veličiny

Označení průřezu	A	I_y	W_y	i_y	I_z	W_z	i_z
IE	mm²	mm⁴	mm³	mm	mm⁴	mm³	mm
80	610	0,81	20,2	32,3	74,9	3,06	9,8
100	1 200	1,98	39,7	40,6	179	6,49	12,2
120	1 470	3,50	58,4	48,8	279	8,72	13,8
140	1 740	5,72	81,7	57,3	419	11,5	15,5
160	2 020	8,73	109	65,7	586	14,5	17,0
180	2 340	12,9	143	74,2	826	18,4	18,8
200	2 680	18,4	184	82,8	1 150	23,1	20,7
220	3 060	25,5	232	91,3	1 570	28,6	22,7
240	3 480	34,6	289	99,7	1 980	34,5	23,9
270	4 020	50,1	371	112	2 600	41,5	25,4
300	4 650	70,8	472	122	3 370	49,9	26,9
330	5 380	98,4	597	135	4 190	59,9	27,9
360	6 190	133,8	743	147	5 160	71,1	28,9
400	7 140	189,3	947	163	6 660	85,9	30,5
450	8 300	274,5	1 220	182	8 070	101	31,2
500	9 780	392,9	1 570	200	10 400	122	32,6
Násobitel	–	10^6	10^3	–	10^3	10^3	–

Moment of inertia
Mixed moment of inertia

Example: Determine the relation for the moment of inertia w.r.t. centroid axes



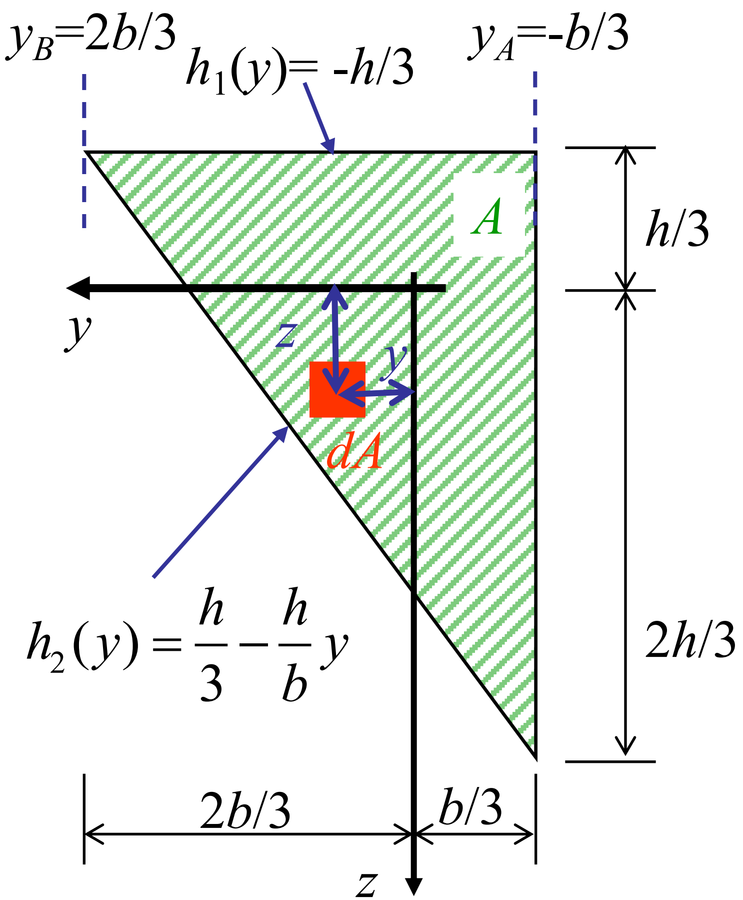
$$I_y = \int_A z^2 dA = \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} z^2 \, dz \, dy$$

$$= \int_{-b/2}^{b/2} \left[\frac{z^3}{3} \right]_{z=-h/2}^{h/2} dy = \int_{-b/2}^{b/2} \frac{h^3}{12} dy$$

$$= \left[\frac{h^3}{12} y \right]_{y=-b/2}^{b/2} = \frac{1}{12} bh^3$$

Moment of inertia
Mixed moment of inertia

Example: Determine the relation for the mixed moment of inertia w.r.t. centroid axis



$$D_{yz} = \int_A yz\, dA = \int_{y=y_A}^{y_B} \left(\int_{z=h_1(y)}^{h_2(y)} yz\, dz \right) dy$$

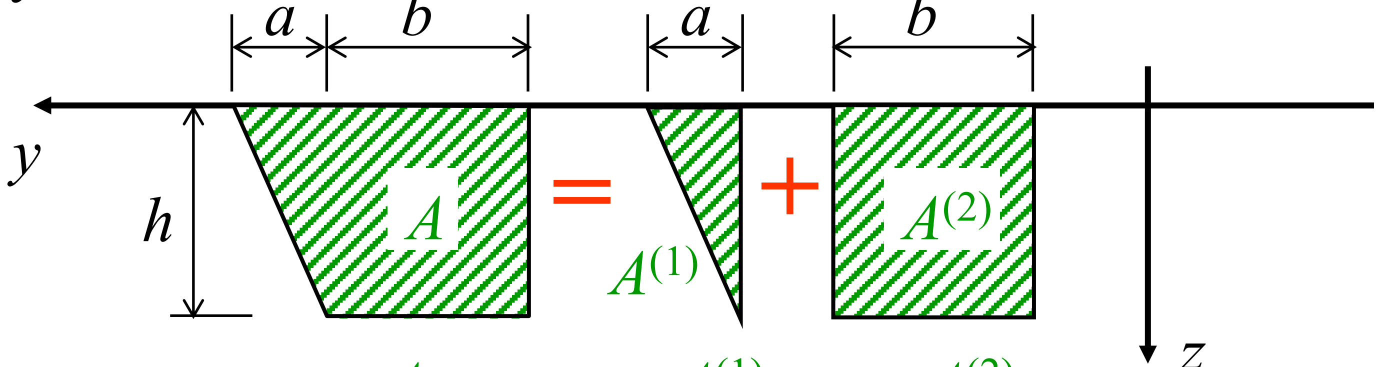
$$= \int_{y=-\frac{b}{3}}^{\frac{2b}{3}} \left(\int_{z=-\frac{h}{3}}^{\frac{h}{3}-\frac{h}{b}y} yz\, dz \right) dy = \int_{y=-\frac{b}{3}}^{\frac{2b}{3}} \left(y \left[\frac{z^2}{2} \right]_{z=-\frac{h}{3}}^{\frac{h}{3}-\frac{h}{b}y} \right) dy$$

$$= \int_{y=-\frac{b}{3}}^{\frac{2b}{3}} y \left(\frac{\left(\frac{h}{3} - \frac{h}{b}y \right)^2}{2} - \frac{\left(-\frac{h}{3} \right)^2}{2} \right) dy$$

$$= \left[-\frac{h^2 y^3}{9b} + \frac{h^2 y^4}{8b^2} \right]_{y=-\frac{b}{3}}^{\frac{2b}{3}} = -\frac{b^2 h^2}{72}$$

Moment of inertia
Mixed moment of inertia

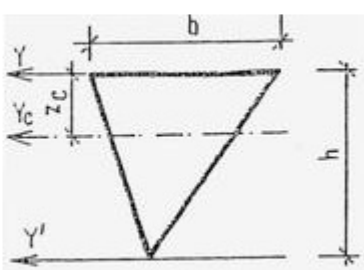
Example: Determine the relation for the moment of inertia w.r.t. a given y-axis



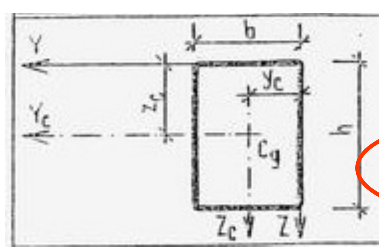
$$A \quad = \quad A^{(1)} \quad + \quad A^{(2)}$$

$$I_y = \int_A z^2 dA = \int_{A^{(1)}} z^2 dA + \int_{A^{(2)}} z^2 dA = I_y^{(1)} + I_y^{(2)}$$ (split of the integral)

Tabulated values:



$$I_{Y_c} = \frac{bh^3}{36}$$
$$I_Y = \frac{bh^3}{12}$$
$$I_{Y'} = \frac{bh^3}{4}$$

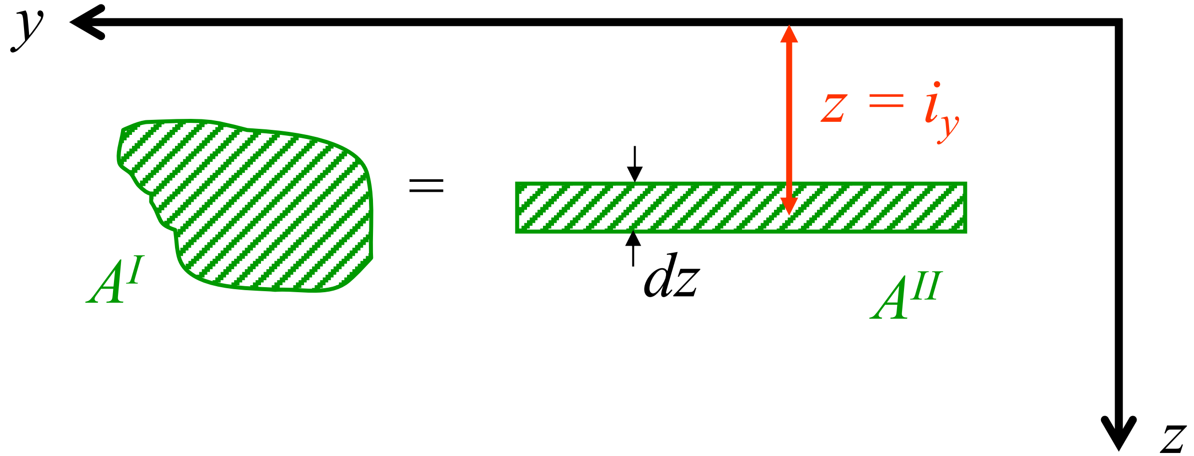


$$I_{Y_c} = \frac{bh^3}{12},$$
$$I_Y = \frac{bh^3}{3},$$

MoI must be w.r.t. the same axes!!!

$$\boxed{I_y = \frac{ah^3}{12} + \frac{bh^3}{3}}$$

Radius of gyration



Assumption:

$$A^I = A^{II} = A$$

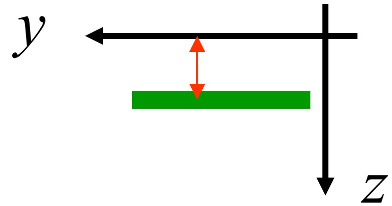
$$I_y^I = I_y^{II} = I_y$$

It is valid that: $I_y^{II} = \int_{A^{II}} z^2 \, dA = i_y^2 \cdot A^{II}$ $\quad (z = \text{const.} = i_y)$

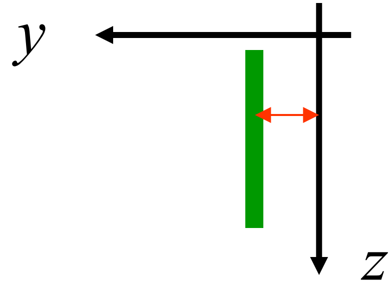
$$\Rightarrow \boxed{I_y = i_y^2 \cdot A}$$

Radius of gyration

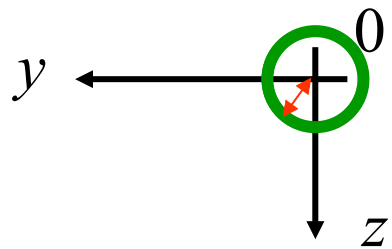
- y axis: $i_y = \sqrt{\dfrac{I_y}{A}}$



- z axis: $i_z = \sqrt{\dfrac{I_z}{A}}$



- w.r.t. 0 : $i_0 = \sqrt{\dfrac{I_0}{A}}$



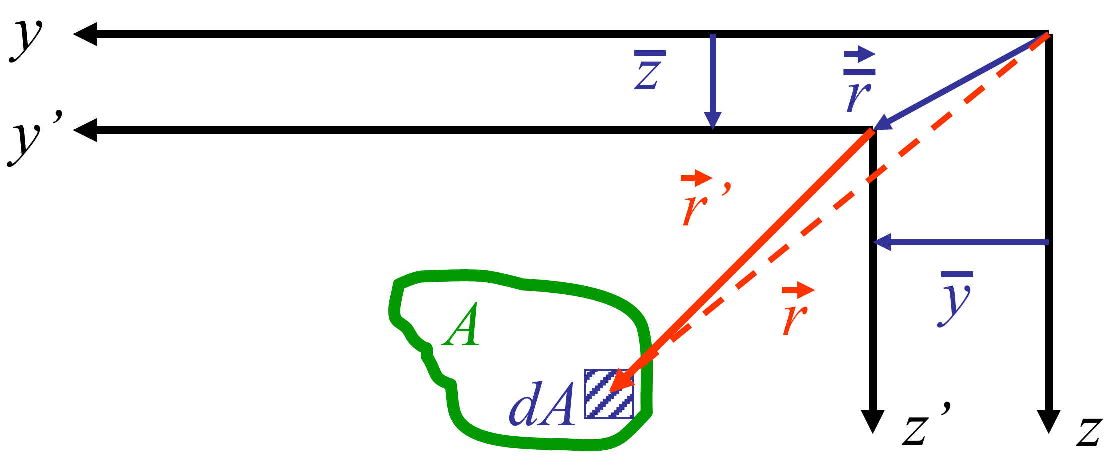
Note: i_y, i_z, i_0 …. [m]

$$i_0^2 = i_y^2 + i_z^2$$

Transformation – translation

Moment of inertia w.r.t. **translated axes**

Moments of inertia and mixed moment of inertia w.r.t. *y*, *z* axis are known, determine these moments w.r.t. *y'*, *z'* axes.



Transformation of coordinates

$$y' = y - \bar{y}$$

$$z' = z - \bar{z}$$

$$\left(\vec{r}\,' = \vec{r} - \vec{\bar{r}}\right)$$

$$I_{z'} = \int_A y'^2\, dA = \int_A (y - \bar{y})^2\, dA = \int_A y^2 dA - 2\bar{y}\int_A y\,dA + \bar{y}^2 \int_A dA$$

$$= I_z - 2\bar{y}S_z + \bar{y}^2 A$$

Note: $\bar{y}$ = const.

Transformation – translation

Using similar approach we get:

$$I_{y'} = I_y - 2\,\overline{z}\,S_y + \overline{z}^{\,2} A$$

$$I_{0'} = I_0 - 2\,\overline{y}\,S_z - 2\,\overline{z}\,S_y + A\left(\overline{y}^{\,2} + \overline{z}^{\,2}\right)$$

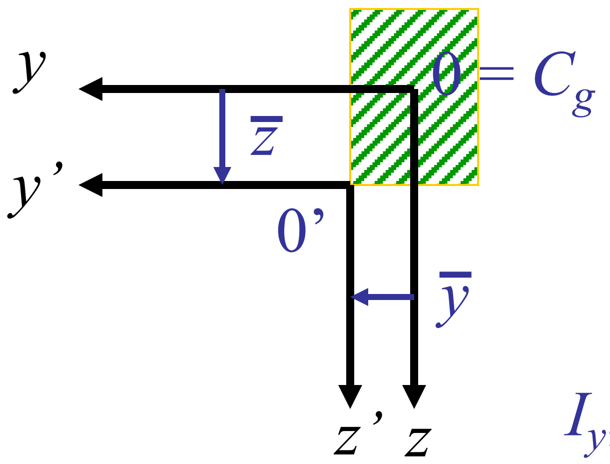
$$D_{y'z'} = D_{yz} - \overline{y}\,S_y - \overline{z}\,S_z + \overline{y}\,\overline{z}\,A$$

Note:

$$I_{0'} = I_{y'} + I_{z'}$$

Transformation – translation (ver. 1)

If the origin of CS is placed into the center of gravity then $S_y = S_z = 0$



$$\left(\begin{array}{l} y_c = \dfrac{S_z}{A} = 0 \\ z_c = \dfrac{S_y}{A} = 0 \end{array} \right)$$

I_y, I_z, D_{yz}, I_0 …. moments of inertia and mixed moment of inertia w.r.t. **centroid CS**

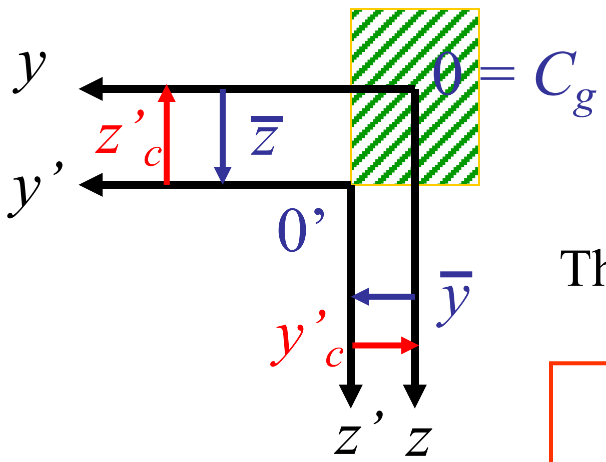
$$I_{y'} = I_y + \overline{z}^2 A \qquad D_{y'z'} = D_{yz} + \overline{y}\,\overline{z}\,A$$

$$I_{z'} = I_z + \overline{y}^2 A \qquad I_{0'} = I_0 + \left(\overline{y}^2 + \overline{z}^2\right) A$$

(Steiner's theorem)

Transformation – translation (ver. 2)

If the origin of CS is placed into the center of gravity then $S_y = S_z = 0$



If:

$$y'_c = -\bar{y}$$
$$z'_c = -\bar{z}$$

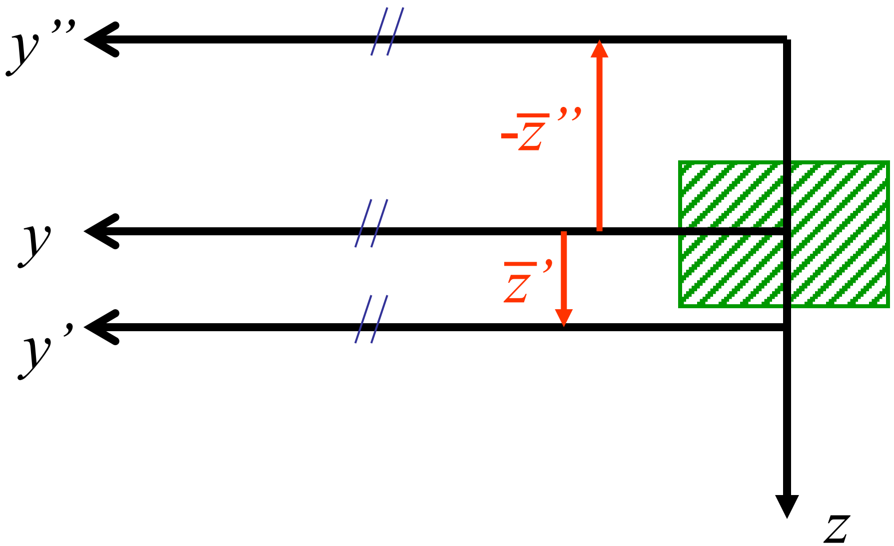
Then the Steiner theorem can be also written as:

$$I_{y'} = I_y + \bar{z}^2 A = I_y + {z'}_c^2 A$$
$$I_{z'} = I_z + \bar{y}^2 A = I_z + {y'}_c^2 A$$
$$D_{y'z'} = D_{yz} + \bar{y}\,\bar{z} A = D_{yz} + y'_c\, z'_c A$$
$$I_{0'} = I_0 + (\bar{y}^2 + \bar{z}^2) A = I_0 + ({y'}_c^2 + {z'}_c^2) A$$

(Steiner's theorem)

Transformation – translation

Note:



$$I_{y'} = I_y + \bar{z}'^2 A$$

$$I_{y''} = I_y + (-\bar{z}'')^2 A$$

$$|\bar{z}'| < |\bar{z}''| \Rightarrow$$

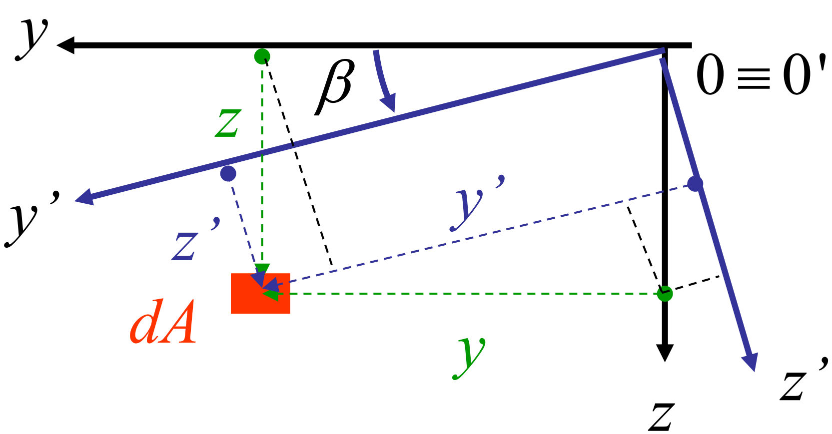
$$I_y < I_{y'} < I_{y''}$$

…. the moment of inertia w.r.t. centroid axis has **the smallest value**.

Transformation – rotation

Moment of inertia w.r.t. **rotated axes**

Moments of inertia and mixed moment of inertia w.r.t. *y*, *z* axis are known, determine these moments w.r.t. *y'*, *z'* axes.



Transformation of coordinates:

$$y' = y \cdot \cos\beta + z \cdot \sin\beta$$

$$z' = -y \cdot \sin\beta + z \cdot \cos\beta$$

$$\{r'\} = \{T\}\{r\}$$

$$I_{z'} = \int_A y'^2\, dA = \int_A \left(y \cdot \cos\beta + z \cdot \sin\beta\right)^2 dA$$

$$= \cos^2\beta \cdot \overbrace{\int_A y^2 dA}^{I_z} + \underbrace{2\sin\beta\cos\beta}_{\sin 2\beta} \overbrace{\int_A yz\,dA}^{D_{yz}} + \sin^2\beta \overbrace{\int_A z^2 dA}^{I_y}$$

$$= I_y \sin^2\beta + I_z \cos^2\beta + D_{yz} \sin 2\beta$$

Transformation – rotation

Using similar approach we get:

$$I_{y'} = I_y \cos^2 \beta + I_z \sin^2 \beta - D_{yz} \sin 2\beta$$

$$D_{y'z'} = \frac{1}{2}\left(I_y - I_z\right)\sin 2\beta + D_{yz} \cos 2\beta$$

Note: this transformation is valid for any CS (not only centroid CS)

Transformation – rotation

$$I_{O'} = \underline{I_{y'} + I_{z'}} = (I_y + I_z)(\cos^2\beta + \sin^2\beta) + \sin 2\beta\left(D_{yz} - D_{yz}\right)$$

$$= \underline{I_y + I_z} = I_O$$

Matrix notation:

$$\begin{bmatrix} I_{y'} & -D_{y'z'} \\ -D_{y'z'} & I_{z'} \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} I_y & -D_{yz} \\ -D_{yz} & I_z \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$[I'] = [T][I][T]^T$$

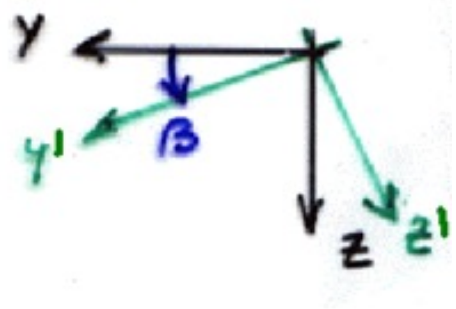
$[I'], [I]$ … inertia tensor

$[T]$ … transformation tensor

Transformation – rotation

Invariants of inertia tensor

Invariant = its value does not change when arbitrary rotations are applied



Linear invariant:

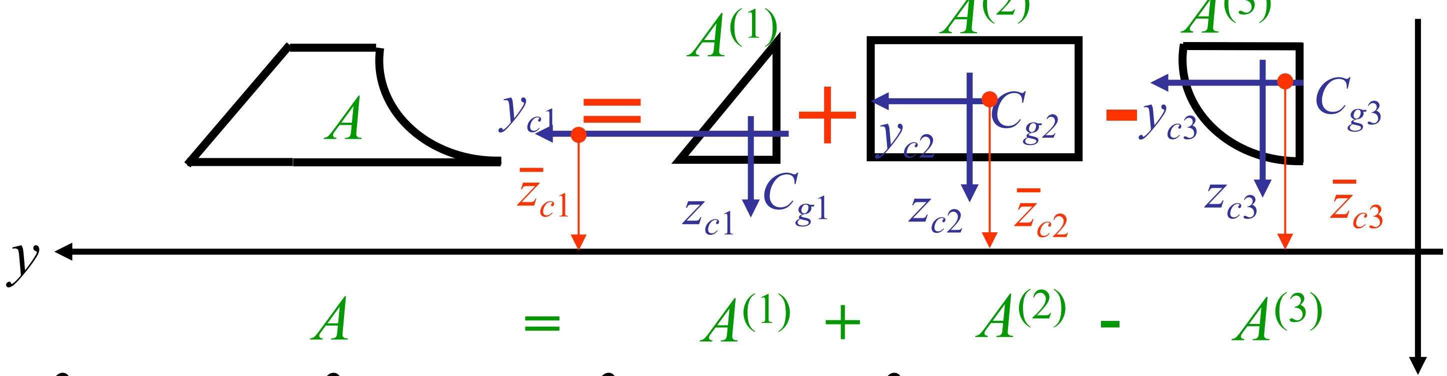
$$I_{1I} = I_y + I_z = I_{y\prime} + I_{z\prime} = I_O$$

Quadratic invariant:

$$I_{2I} = I_y I_z - D_{yz}^2 = I_{y'} I_{z'} - D_{y'z'}^2$$

Transformation – example

Determine the moments of inertia of **complex shapes**



$$A = A^{(1)} + A^{(2)} - A^{(3)}$$

$$I_y = \int_A z^2 dA = \int_{A^{(1)}} z^2 dA + \int_{A^{(2)}} z^2 dA - \int_{A^{(3)}} z^2 dA$$

$$= I_y^{(1)} + I_y^{(2)} - I_y^{(3)}$$ (Note: all w.r.t. y axis!)

$$I_y^{(1)} = \underline{I_{yc1}^{(1)}} + A^{(1)}\underline{\bar{z}_{c1}^2}$$

$$I_y^{(2)} = \underline{I_{yc2}^{(2)}} + A^{(2)}\underline{\bar{z}_{c2}^2}$$

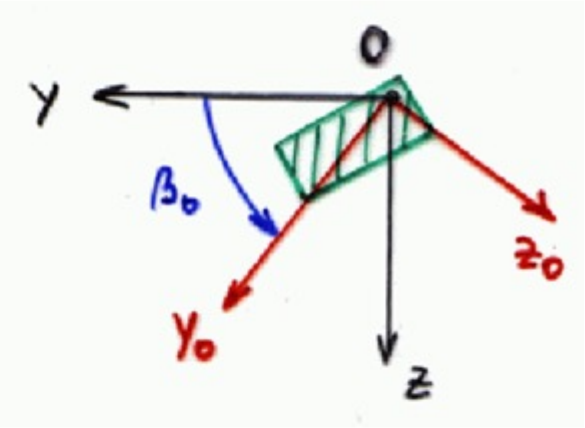
$$I_y^{(3)} = \underline{I_{yc3}^{(3)}} + A^{(3)}\underline{\bar{z}_{c3}^2}$$

Coordinates of origin of y, z axes (w.r.t y_{ci}, z_{ci} axes)

Moments of inertia w.r.t. centroids of individual areas; e.g. tabulated value

Moment of inertia – principal axes

For a given origin "O" we seek the extreme values of moments of inertia => **principal axis**



$$\frac{\mathrm{d}I_{y'}}{\mathrm{d}\beta} = \left(I_z - I_y\right)\sin 2\beta - 2D_{yz}\cos 2\beta$$

$$= -\frac{\mathrm{d}I_{z'}}{\mathrm{d}\beta}$$

$$\frac{\mathrm{d}I_{y'}}{\mathrm{d}\beta} = 0 \Rightarrow \tan 2\beta_0 = \frac{2D_{yz}}{I_z - I_y}$$

$$\frac{\mathrm{d}I_{z'}}{\mathrm{d}\beta} = 0 \quad \Nearrow$$

principal axis of inertia

Moment of inertia – principal axes

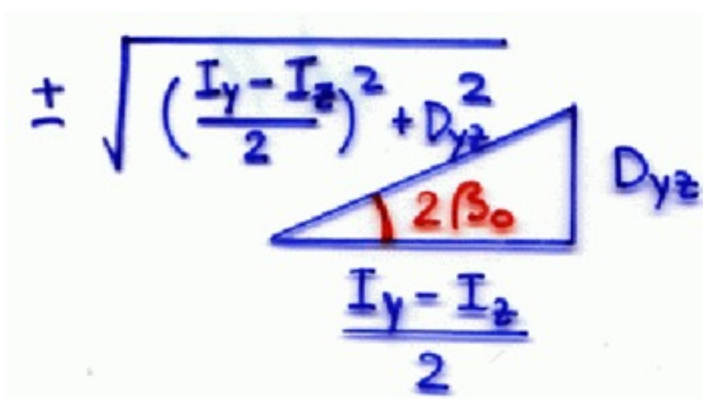
Principal moments of inertia

$$I_{y0} = I_y \cos^2\beta_0 + I_z \sin^2\beta_0 - D_{yz}\sin 2\beta_0$$

$$I_{z0} = I_y \sin^2\beta_0 + I_z \cos^2\beta_0 + D_{yz}\sin 2\beta_0$$

$$\cos^2\beta_0 = \frac{1}{2}(\cos 2\beta_0 + 1);\ \sin^2\beta_0 = \frac{1}{2}(1 - \cos 2\beta_0);$$

$$I_{y0} = \frac{I_y + I_z}{2} + \frac{I_y - I_z}{2}\cos^2\beta_0 - D_{yz}\sin 2\beta_0$$



$$\cos 2\beta_0 = \frac{I_y - I_z}{2\sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + D_{yz}^2}}$$

$$\sin 2\beta_0 = \ \ldots$$

Moment of inertia – principal axes

$$I_{y0} = \frac{I_y + I_z}{2} \pm \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + D_{yz}^2}$$

$$I_{z0} = \frac{I_y + I_z}{2} \mp \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + D_{yz}^2}$$

Principal moments of inertia $I_{1,2}$ $(I_1 \geq I_2)$

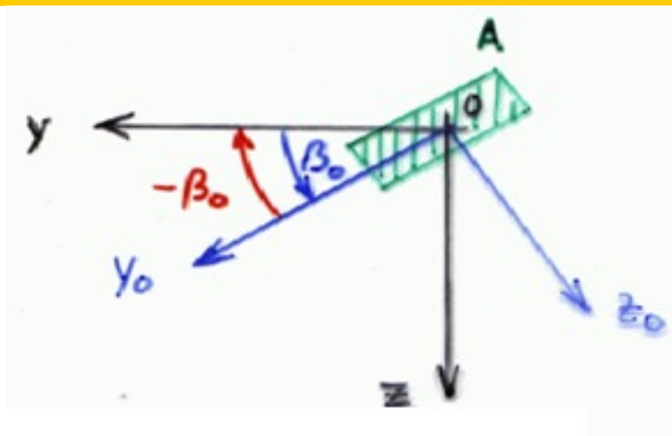
$$D_{y0z0} = 0$$

Note:

$I_y + I_z$ … first invariant

$\sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + D^2{}_{yz}}$ … must be independent on the rotation of CS

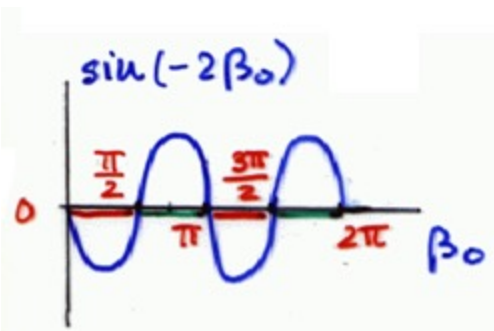
Moment of inertia – principal axes

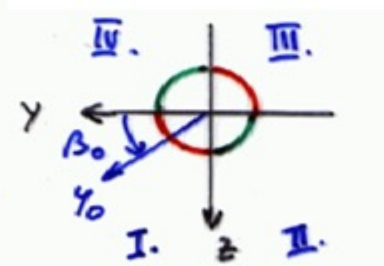


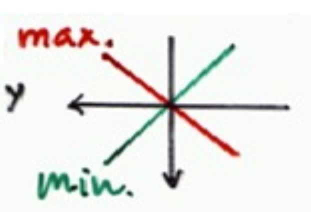
Transformation $(y_0, z_0) \rightarrow (y, z)$

$$D_{yz} = \frac{1}{2}(I_{y_0} - I_{z_0}) \sin(-2\beta_0)$$

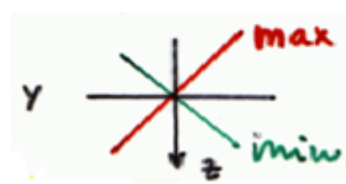
>0	$I_{y_0} > I_{z_0}$	>0	II., IV.
>0	$I_{z_0} > I_{y_0}$	<0	I., III.
<0	$I_{z_0} > I_{y_0}$	>0	II., IV.
<0	$I_{y_0} > I_{z_0}$	<0	I., III.
	↑ I_1 ↑ I_2		





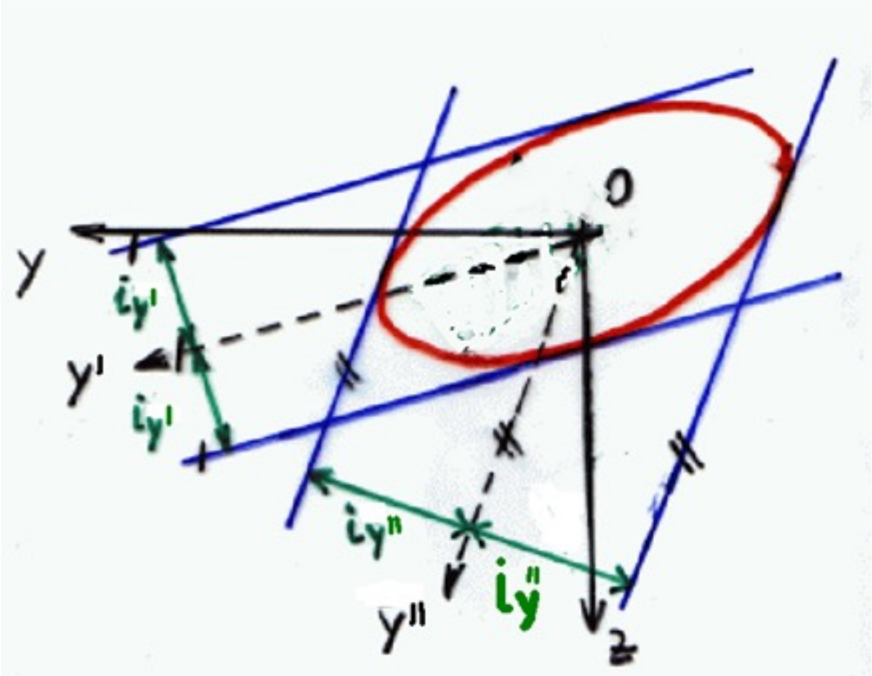


$D_{yz} > 0 \Rightarrow$ axis of max. moment is in interval II. and IV.



$D_{yz} < 0 \Rightarrow$ axis of max. moment is in interval I. and III.

Ellipse of inertia



Lines with the distance equal to the corresponding radius of gyration $(i_y, i_{y'}, i_{y''}, \ldots)$ are drawn parallel to the axis of inertia

These lines are tangents to the **ellipse of inertia**

Note:

$$i_{y'} = \sqrt{\frac{I_{y'}}{A}}$$

Ellipse of inertia

Equation of ellipse of inertia:

$$I_y y^2 + I_z z^2 - 2D_{yz} yz = \frac{I_y I_z - D_{yz}^2}{A}$$

This equation can be written in the form:

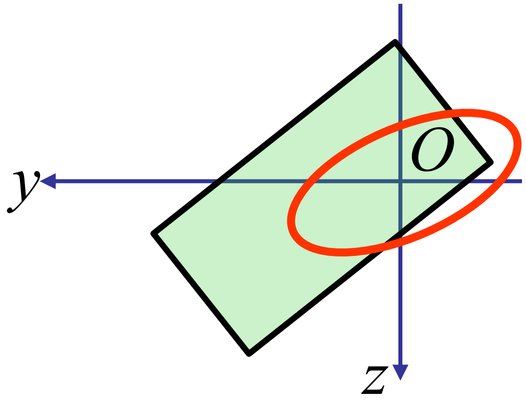
$$\{y, z\} \begin{bmatrix} I_y & -D_{yz} \\ -D_{yz} & I_z \end{bmatrix} \begin{Bmatrix} y \\ z \end{Bmatrix} = \frac{I_y I_z - D_{yz}^2}{A}$$

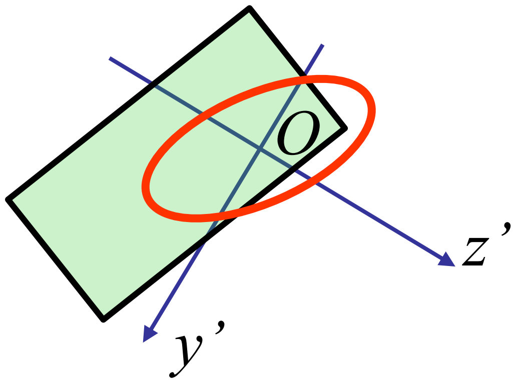
I_{20} **invariant**

Ellipse of inertia

It can be shown by means of the transformation relations that for two coordinate systems rotated again each other we can write:

$$\{y,z\}\begin{bmatrix} I_y & -D_{yz} \\ -D_{yz} & I_z \end{bmatrix}\begin{Bmatrix} y \\ z \end{Bmatrix} = \{y',z'\}\begin{bmatrix} I_{y\prime} & -D_{y\prime z\prime} \\ -D_{y\prime z\prime} & I_{z\prime} \end{bmatrix}\begin{Bmatrix} y' \\ z' \end{Bmatrix}$$

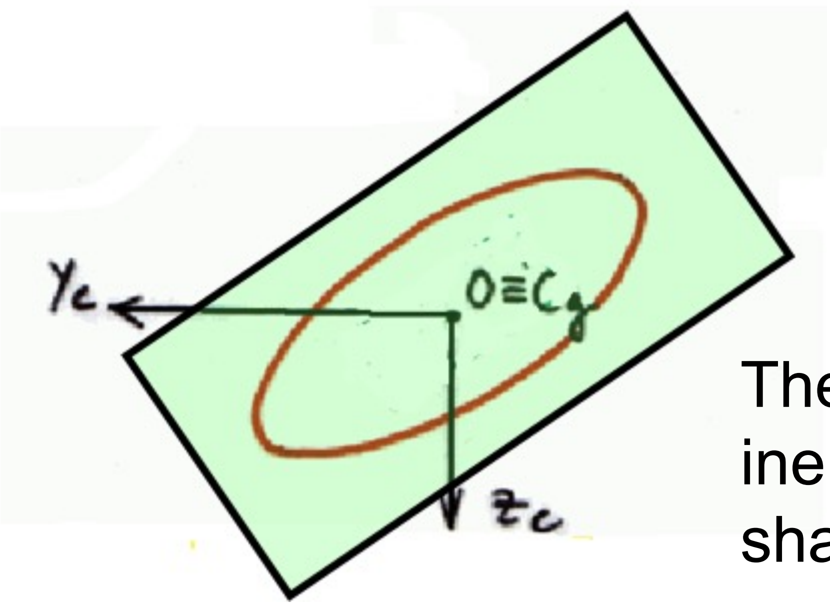




Therefore, the shape of ellipse of inertia does not depend on the rotation of coordinate system but only on the shape of the cross-section and the position of origin "O"

Central ellipse of inertia

If the origin of CS coincides with the center of gravity
=> **central ellipse of inertia**



The shape of ellipse of inertia partly follows the shape of cross-section

Central ellipse of inertia

Equation of central ellipse of inertia:

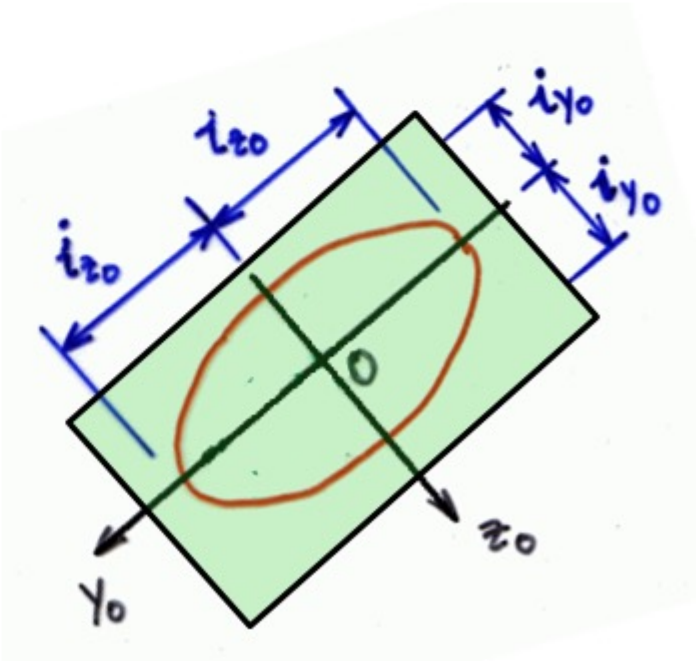
$$I_{y0} y_0^2 + I_{z0} z_0^2 - 0 = \frac{I_{y0} I_{z0} - 0}{A} \quad \Big/ \cdot \frac{A}{I_{y0} I_{z0}}$$

$$\frac{A}{I_{z0}} y_0^2 + \frac{A}{I_{y0}} z_0^2 = 1$$

$$\frac{y_0^2}{i_{z0}^2} + \frac{z_0^2}{i_{y0}^2} = 1$$

i_{y0}, i_{z0} … ellipse semi-axes

y_0, z_0 … principal axes of ellipse of inertia



The End