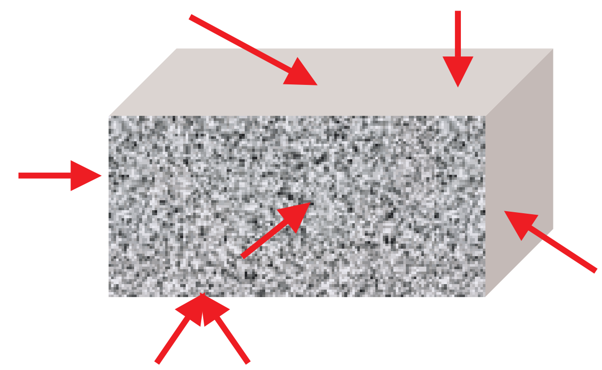
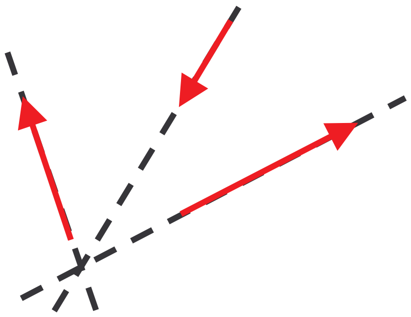
3.3 System of forces

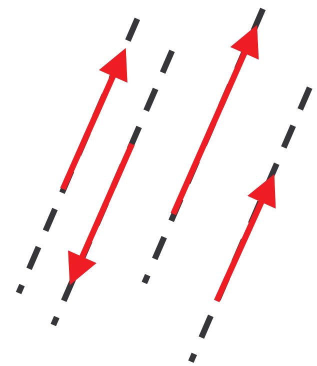


- system of forces = set of forces action on a body

- special cases:
  - concurrent system of forces (lines of action of all forces intersect at one point)

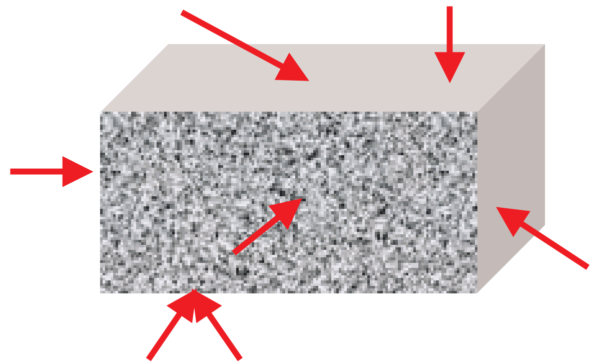


  - system of parallel forces (lines of action of all forces are parallel to each other)



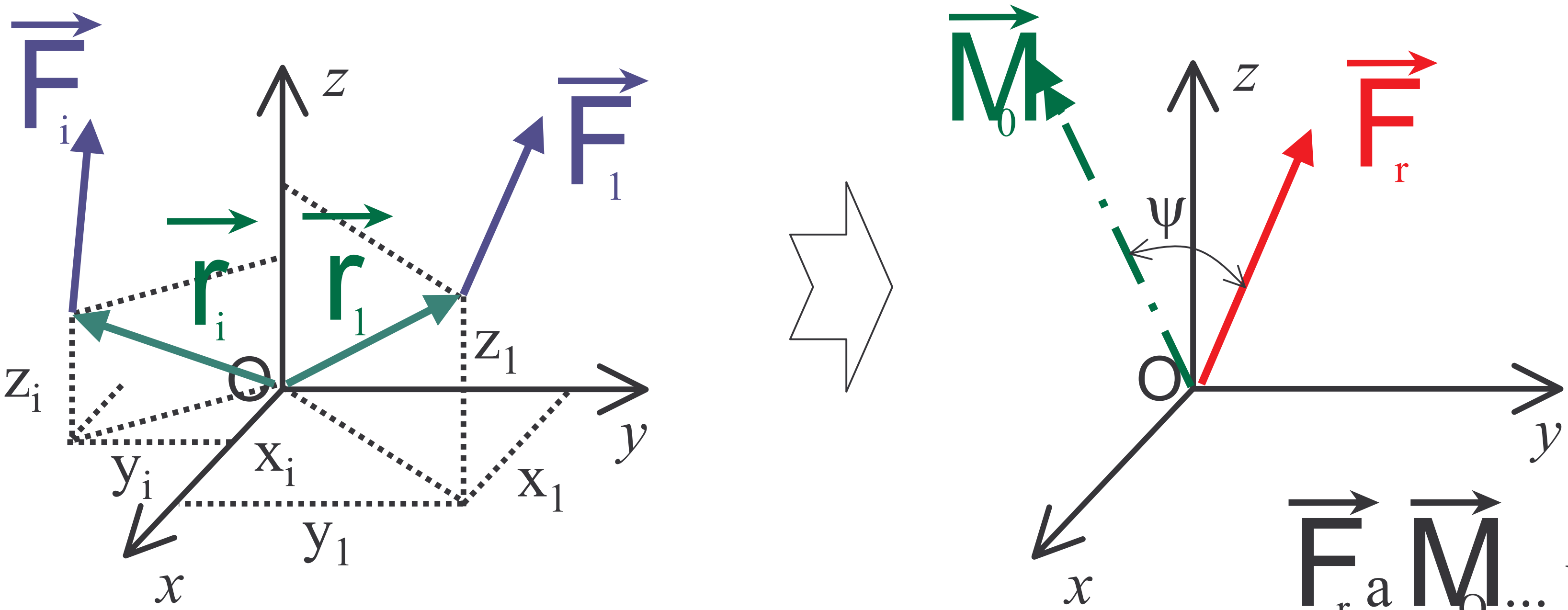
  - plane system of forces (lines of action of all forces are found in a single plane)
    - plane system of concurrent forces
    - plane system of parallel forces

3.3.1 General system of forces



Resulting effect

- resulting effect of a system of *n* forces $\{\vec{F}_i\}$ is determined by reduction of individual forces w.r.t the c.s. origin and by vector sum of reduced forces and static moments



$\vec{F}_r$ a $\vec{M}_O$... **bivector**

$$\vec{F}_r = \sum_{i=1}^{n} \vec{F}_i \qquad \vec{M}_O = \sum_{i=1}^{n} \vec{M}_{Oi} = \sum_{i=1}^{n} (\vec{r}_i \times \vec{F}_i)$$

Components:

$$F_{rx} = \sum_{i=1}^{n} F_{ix} = \sum_{i=1}^{n} F_i \cos\alpha_i \qquad M_x = \sum_{i=1}^{n} M_{ix} = \sum_{i=1}^{n} \left(r_{iy}F_{iz} - F_{iy}r_{iz}\right)$$

$$F_{ry} = \sum_{i=1}^{n} F_{iy} = \sum_{i=1}^{n} F_i \cos\beta_i \qquad M_y = \sum_{i=1}^{n} M_{iy} = \sum_{i=1}^{n} \left(r_{iz}F_{ix} - F_{iz}r_{ix}\right)$$

$$F_{rz} = \sum_{i=1}^{n} F_{iz} = \sum_{i=1}^{n} F_i \cos\gamma_i \qquad M_z = \sum_{i=1}^{n} M_{iz} = \sum_{i=1}^{n} \left(r_{ix}F_{iy} - F_{ix}r_{iy}\right)$$

- Magnitude of $\vec{F}_r$:

$$F_r = (F_{rx}^2 + F_{ry}^2 + F_{rz}^2)^{1/2}$$

- Magnitude of $\vec{M}_0$:

$$M_0 = (M_x^2 + M_y^2 + M_z^2)^{1/2}$$

- directional cosines:

$$\cos\alpha_r = \frac{F_{rx}}{F_r} \quad \cos\beta_r = \frac{F_{ry}}{F_r} \quad \cos\gamma_r = \frac{F_{rz}}{F_r} \qquad \cos\lambda = \frac{M_{0x}}{M_0} \quad \cos\mu = \frac{M_{0y}}{M_0} \quad \cos\nu = \frac{M_{0z}}{M_0}$$

Mutual angle between vectors $\vec{F}_r$ and $\vec{M}_O$: ψ

$$\vec{F}_r \cdot \vec{M}_O = F_r M_O \cos\psi = F_{rx} M_x + F_{ry} M_y + F_{rz} M_z$$

$$\cos\psi = \frac{1}{F_r M_0}(F_{rx} M_x + F_{ry} M_y + F_{rz} M_z)$$

$$= \cos\alpha_r \cos\lambda + \cos\beta_r \cos\mu + \cos\gamma_r \cos\nu$$

In general $\cos\psi \neq 0$

Special cases:

- $\vec{F}_r \cdot \vec{M}_O = 0$, $\vec{F}_r$ and $\vec{M}_O$ are perpendicular to each other $\Rightarrow$ the resultant is a single forces $\vec{F}_r$ acting along a certain line of action such that its static moment taken w.r.t. the c.s. origin $\vec{M}_O$ zero. The expressions

$$F_{rz}y - F_{ry}z = M_x \qquad F_{rx}z - F_{rz}x = M_y \qquad F_{ry}x - F_{rx}y = M_z$$

then determine the equation of the line of action of $\vec{F}_r$

- $\vec{F}_r \neq \vec{0}$, $\vec{M}_O = \vec{0} \Rightarrow$ the resultant is a single force $\vec{F}_r$ acting along the line of action that passes through the c.s. origin
- $\vec{F}_r = \vec{0}$, $\vec{M}_O \neq \vec{0} \Rightarrow$ the resultant is a couple force acting in the plane normal to the line of action of $\vec{M}_O$ and giving the static moment $M_d = M_O$
- $\vec{F}_r = \vec{0}$, $\vec{M}_O = \vec{0} \Rightarrow$ the system of forces is in equilibrium

Equations of equilibrium

System of n forces $\{\vec{F}_i\}$ is in the state of equilibrium,
if both the resultant force and the resultant moment are equal to zero:

$$\sum_{i=1}^{n} F_{ix} = 0 \qquad \sum_{i=1}^{n} M_{ix} = \sum_{i=1}^{n} \left(F_{iz} y_i - F_{iy} z_i\right) = 0$$

$$\sum_{i=1}^{n} F_{iy} = 0 \qquad \sum_{i=1}^{n} M_{iy} = \sum_{i=1}^{n} \left(F_{ix} z_i - F_{iz} x_i\right) = 0$$

$$\sum_{i=1}^{n} F_{iz} = 0 \qquad \sum_{i=1}^{n} M_{iz} = \sum_{i=1}^{n} \left(F_{iy} x_i - F_{ix} y_i\right) = 0$$

Problem of equilibrium

For a given system of n forces $\{\vec{F}_i\}$ determine the system of m forces $\{\vec{R}_j\}$ that brings the original system of forces into the state of equilibrium

$$\sum_{i=1}^{n} \vec{F}_i + \sum_{j=1}^{m} \vec{R}_j = \vec{0} \qquad \sum_{i=1}^{n} \vec{M}_{OFi} + \sum_{j=1}^{m} \vec{M}_{ORj} = \vec{0}$$

In terms of components:

$$\sum_{i=1}^{n} F_{ix} + \sum_{j=1}^{m} R_{jx} = 0 \qquad \sum_{i=1}^{n} M_{Fix} + \sum_{j=1}^{m} M_{Rjx} = 0$$

$$\sum_{i=1}^{n} F_{iy} + \sum_{j=1}^{m} R_{jy} = 0 \qquad \sum_{i=1}^{n} M_{Fiy} + \sum_{j=1}^{m} M_{Rjy} = 0$$

$$\sum_{i=1}^{n} F_{iz} + \sum_{j=1}^{m} R_{jz} = 0 \qquad \sum_{i=1}^{n} M_{Fiz} + \sum_{j=1}^{m} M_{Rjz} = 0$$

(6 equations - 6 unknowns)

Problem of equivalency

Replace a system of n forces $\{\vec{F}_i\}$ by the system of m forces $\{\vec{R}_j\}$, such that their resultants are the same.

$$\sum_{i=1}^{n} \vec{F}_i = \sum_{j=1}^{m} \vec{R}_j \qquad \sum_{i=1}^{n} \vec{M}_{OFi} = \sum_{j=1}^{m} \vec{M}_{ORj}$$

In terms of components:

$$\sum_{i=1}^{n} F_{ix} = \sum_{j=1}^{m} R_{jx} \qquad \sum_{i=1}^{n} M_{Fix} = \sum_{j=1}^{m} M_{Rjx}$$

$$\sum_{i=1}^{n} F_{iy} = \sum_{j=1}^{m} R_{jy} \qquad \sum_{i=1}^{n} M_{Fiy} = \sum_{j=1}^{m} M_{Rjy}$$

$$\sum_{i=1}^{n} F_{iz} = \sum_{j=1}^{m} R_{jz} \qquad \sum_{i=1}^{n} M_{Fiz} = \sum_{j=1}^{m} M_{Rjz}$$

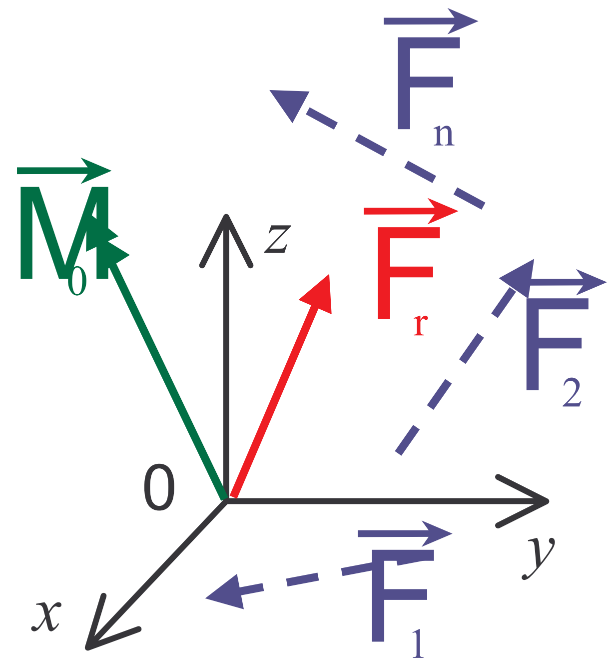
(6 equations - 6 unknowns)

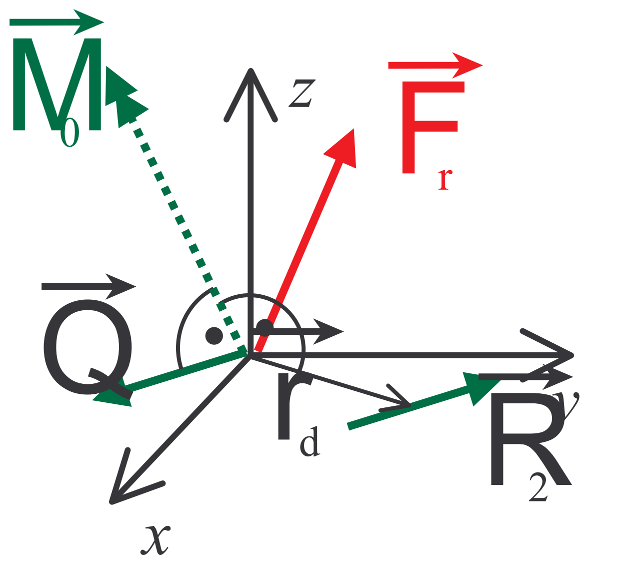
Example problems of equilibrium/equivalency

Ex. 1: Replace a given system of forces $\{\vec{F}_i\}$ by two skew forces $\vec{R}_1$ and $\vec{R}_2$

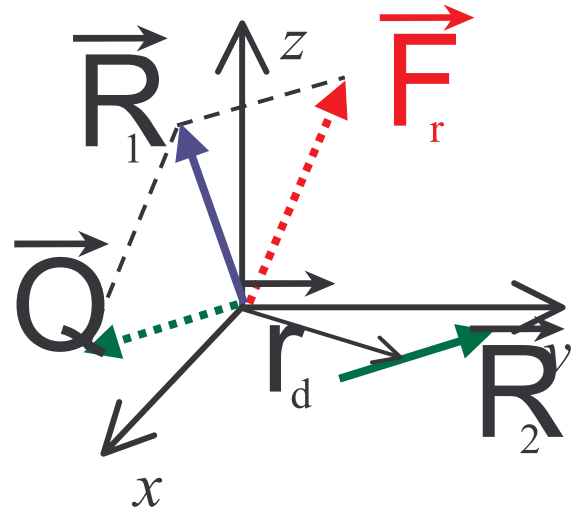
1. The effect of the system $\{\vec{F}_i\}$: can be represented by its resultants $\vec{F}_r, \vec{M}_0$
2. Vector $\vec{M}_0$ is replaced by two forces $\{\vec{R}_2, \vec{Q}\}$ acting in the plane normal to $\vec{M}_0$; $\vec{Q}$ passes through the c.s. origin so that

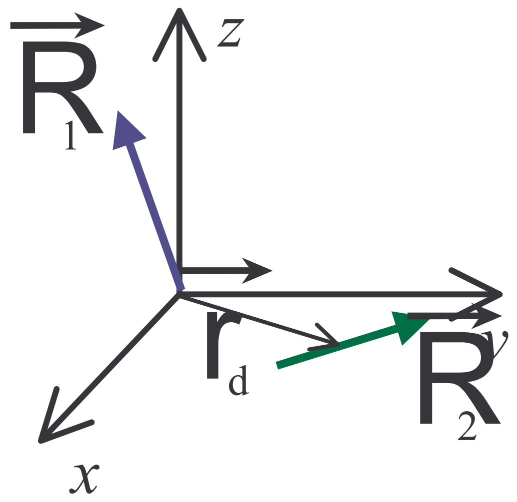
$$\vec{Q} = -\vec{R}_2$$
$$\vec{M}_0 = \vec{r}_d \times \vec{R}_2$$
$$\vec{R}_2 \cdot \vec{M}_0 = 0$$
$$\vec{r}_d \cdot \vec{M}_0 = 0$$





3. Summing $\vec{F}_r + \vec{Q} = \vec{R}_1$

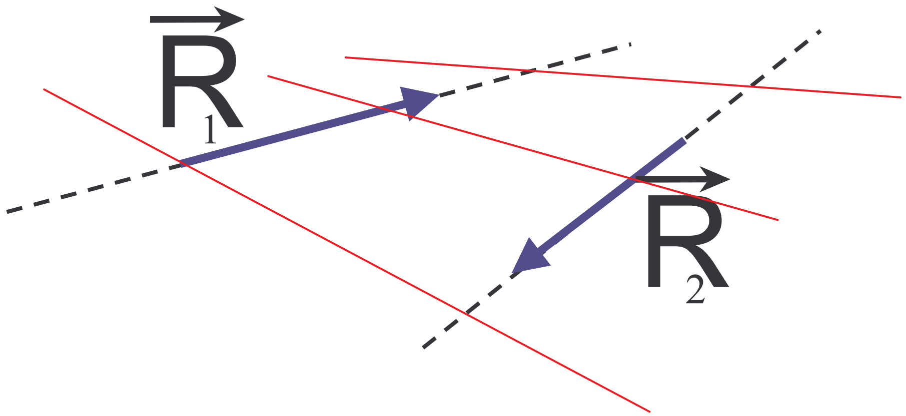




Note:

Zero (nodal) lines of the system of forces = lines about which the static moment is equal to zero

If the system of forces is replaced by to skew forces, then each line that intersects both forces is the nodal line of the original system of forces.



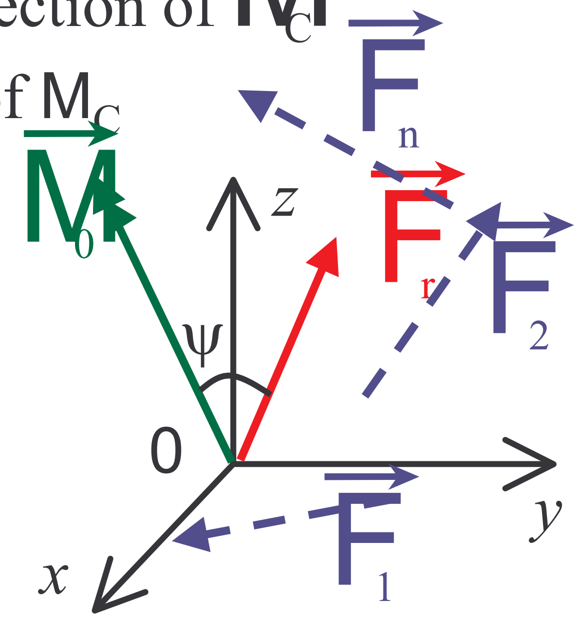
Example problems of equilibrium/equivalency

Ex. 2: Replace the system of forces $\{\vec{F}_i\}$ by a force $\vec{R}_C$ acting along the main axis of the system c and by the main moment $\vec{M}_C$

Definition: $\vec{R}_C \parallel \vec{F}_r$, $R_C = F_r$, $\vec{M}_C$ and $\vec{R}_C$ are coaxial

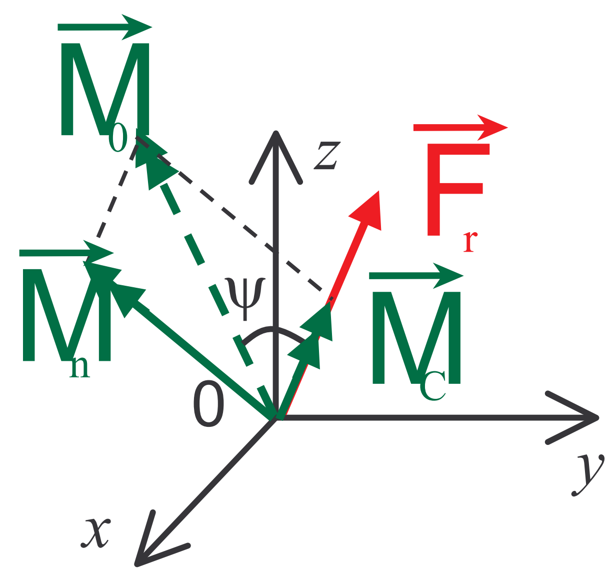
Given: $\{\vec{F}_i\}$, magnitude and direction of $\vec{R}_C$ and direction of $\vec{M}_C$

Determine: the line of action of $\vec{R}_C$ and magnitude of M_C



a) The original system of forces is replaced by the resultant force $\vec{F}_r$ and by the moment $\vec{M}_O$ about the c.s. origin

b) Moment $\vec{M}_O$ is replaced by the moment $\vec{M}_C$ that is coaxial with $\vec{F}_r$ and $\vec{M}_n \perp \vec{F}_r$



$M_C = M_O \cos\psi$, $\quad M_n = M_O \sin\psi$

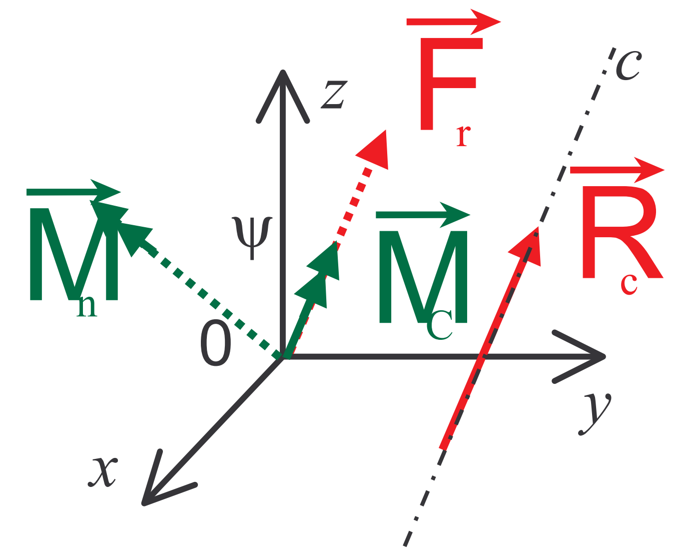
Components of $\vec{M}_C$:

$$M_{Cx} = M_C \cos \alpha_r$$
$$M_{Cy} = M_C \cos \beta_r$$
$$M_{Cz} = M_C \cos \gamma_r$$

Components of $\vec{M}_n = \vec{M}_O - \vec{M}_C$:

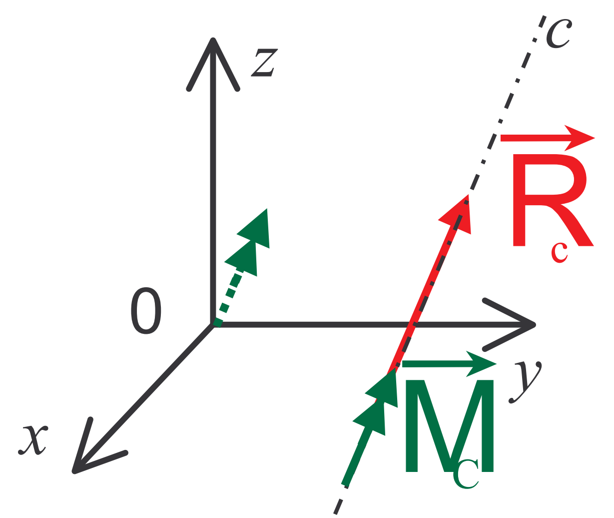
$$M_{nx} = M_{Ox} - M_{Cx}$$
$$M_{ny} = M_{Oy} - M_{Cy}$$
$$M_{nz} = M_{Oz} - M_{Cz}$$

c) Normal vectors $\vec{M}_n$ and $\vec{F}_r$ tied to the origin O are replaced by the forces $\vec{R}_C = \vec{F}_r$ acting along the main line c given by



$$F_{rz}y - F_{ry}z = M_{nx} \qquad F_{rx}z - F_{rz}x = M_{ny} \qquad F_{ry}x - F_{rx}y = M_{nz}$$

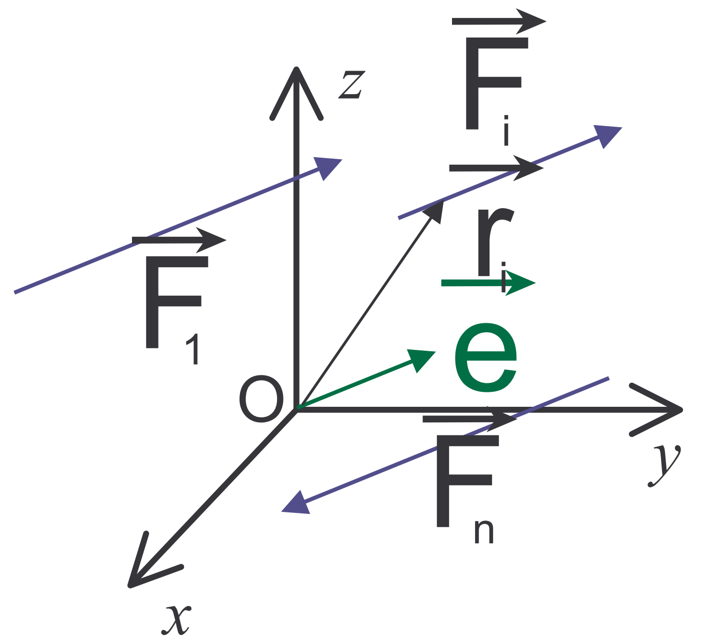
d) Moment $\vec{M}_C$ is shifted to the main axis



The problem of equilibrium could be formulated in the similar manner

3.3.2 General (space) system of parallel forces

Is a special case of a general space system of forces, in which all the lines of action are parallel to each other.



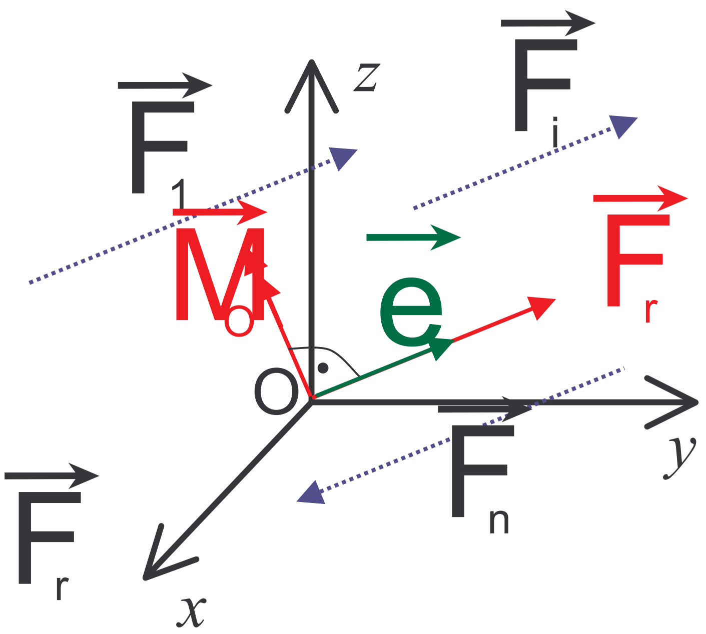
$$\{\vec{F}_i\} = \{F_i \vec{e}\}$$

The resulting effect (w.r.t. the origin O)

$$\vec{F}_r = \sum_{i=1}^{n} \vec{F}_i = \sum_{i=1}^{n} F_i \vec{e} = \vec{e} \sum_{i=1}^{n} F_i = \vec{e} F_r \quad \ldots (\vec{F}_r \parallel \vec{e})$$

$$\vec{M}_O = \sum_{i=1}^{n} \vec{M}_{Oi} = \sum_{i=1}^{n} \left(\vec{r}_i \times \vec{F}_i \right) = \sum_{i=1}^{n} (\vec{r}_i \times F_i \vec{e}) = \sum_{i=1}^{n} (\vec{r}_i F_i) \times \vec{e}$$

... From the definition of cross product $\Rightarrow \vec{M}_O \perp \vec{F}_r$



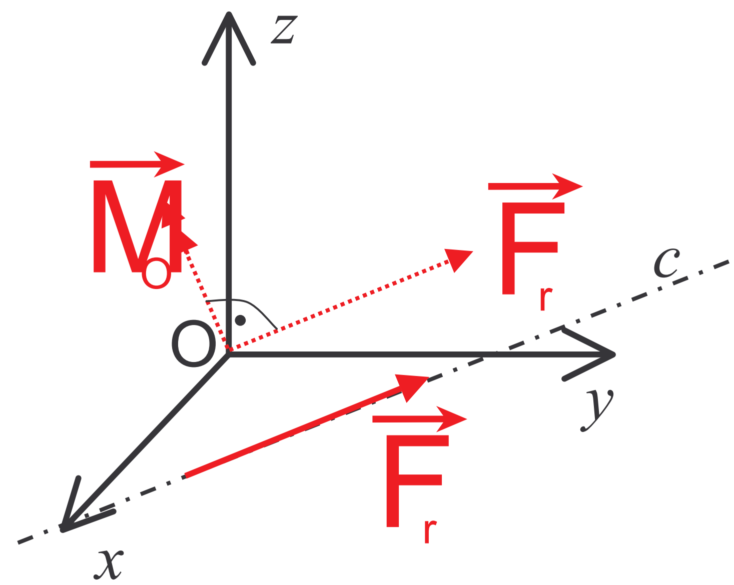
Magnitude of the resultant

$$F_r = \sqrt{F_{rx}^{\ 2} + F_{ry}^{\ 2} + F_{rz}^{\ 2}} = \sqrt{\left(\sum_{i=1}^{n} F_i \right)^2 \left(e_x^2 + e_y^2 + e_z^2 \right)} = \sum_{i=1}^{n} F_i$$

In general $\vec{F}_r \neq \vec{0}$ a $\vec{M}_O \neq \vec{0} \Rightarrow$ the resultant is a single force $\vec{F}_r$ acting along the main axis of the system. Equations

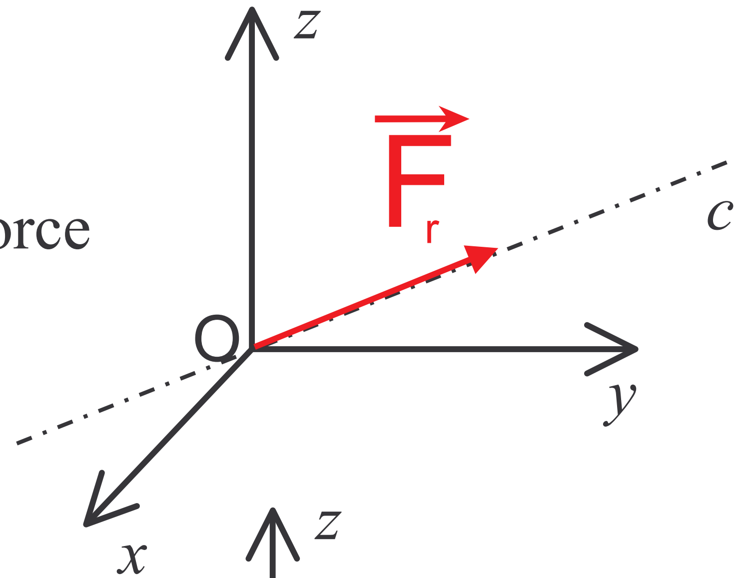
$$F_{rz}y - F_{ry}z = M_x \qquad F_{rx}z - F_{rz}x = M_y \qquad F_{ry}x - F_{rx}y = M_z$$

determine the line of action of $\vec{F}_r$ (the main axis *c*)

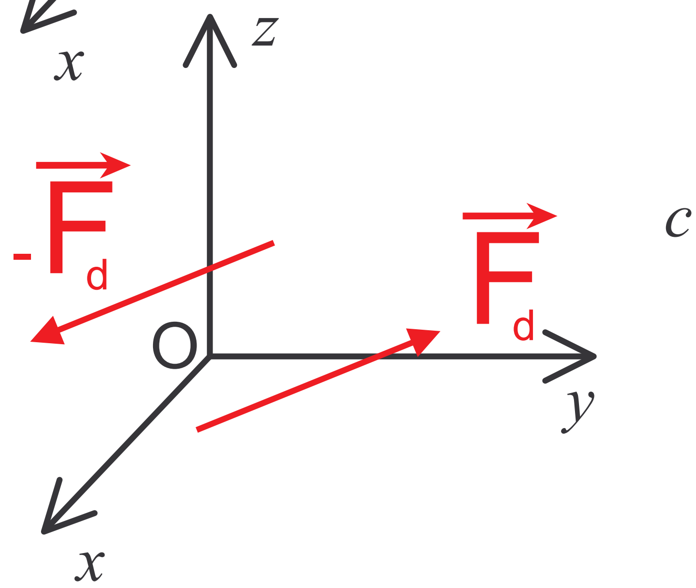


Special cases:

- $\vec{F}_r \neq \vec{0}, \vec{M}_O = \vec{0} \Rightarrow$ the resultant is a single force $\vec{F}_r$ passing through the c.s. origin O

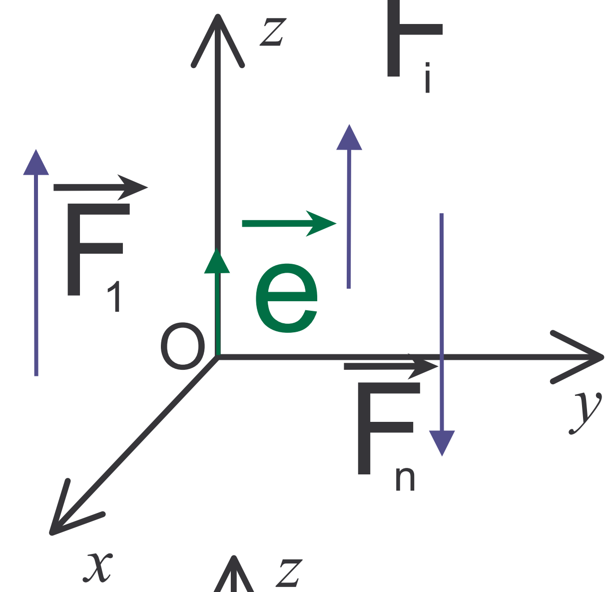


- $\vec{F}_r = \vec{0}, \vec{M}_O \neq \vec{0} \Rightarrow$ the resultant is a force couple acting in the plane normal to the vector $\vec{M}_O$ and giving the moment $M_d = M_O$



- $\vec{F}_r = \vec{0}, \vec{M}_O = \vec{0} \Rightarrow$ the system of forces is in equilibrium

Example: Determine the magnitude and direction of the resultant force $\vec{F}_r$ of the system of forces $\{\vec{F}_i\}$ parallel to the z-axis



For all forces in the system: $F_{ix} = F_{iy} = 0$

$\Rightarrow\ F_{zi} = F_i$

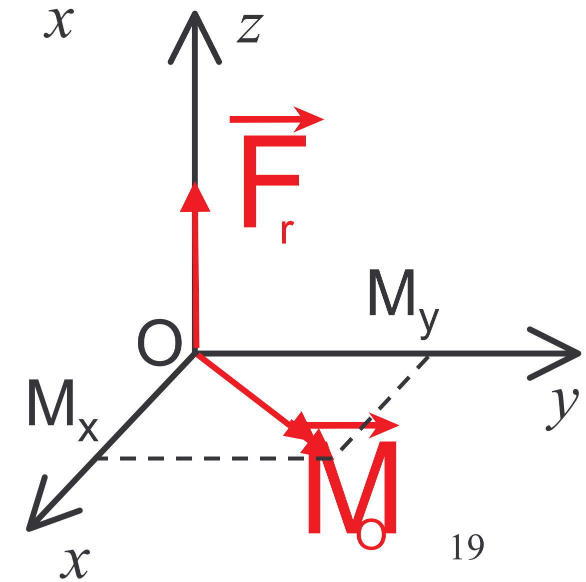
$\Rightarrow\ M_{zi} = 0$

Then: $F_{rx} = F_{ry} = 0, \quad M_z = 0$

$$F_{rz} = \sum_{i=1}^{n} F_{iz} = \sum_{i=1}^{n} F_i$$

$$M_x = \sum_{i=1}^{n} M_{ix} = \sum_{i=1}^{n} \left(F_{iz} y_i - F_{iy} z_i\right) = \sum_{i=1}^{n} F_i y_i$$

$$M_y = \sum_{i=1}^{n} M_{iy} = \sum_{i=1}^{n} \left(F_{ix} z_i - F_{iz} x_i\right) = \sum_{i=1}^{n} (-F_i x_i)$$



Location of the point of application:

$$M_x = \sum_{i=1}^{n} F_i y_i = F_r y_r \Rightarrow y_r = \frac{\sum_{i=1}^{n} F_i y_i}{F_r}$$

$$M_y = \sum_{i=1}^{n} (-F_i x_i) = -F_r x_r \Rightarrow x_r = -\frac{\sum_{i=1}^{n} F_i x_i}{F_r}$$