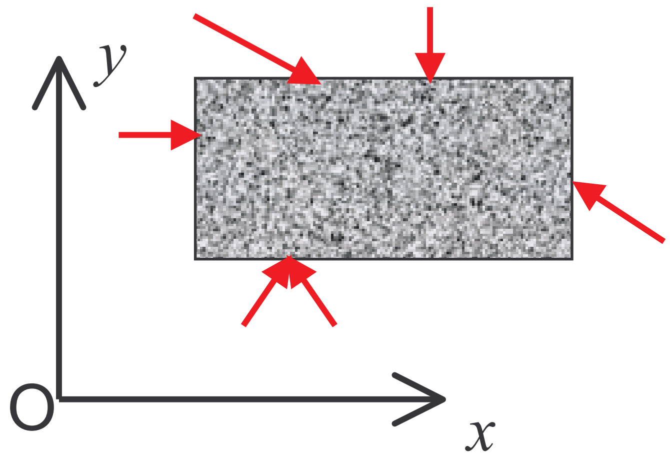
3.3.3 Coplanar system of forces



<u>Assumptions</u>

- All forces are found in a single plane.
- The x-y plane is assumed to contain the system of forces

Coplanar system of forces is a special case of the general system of forces.
Rovinná soustava sil je zvláštním případem prostorové soustavy sil.
$\Rightarrow$ All relations (the resultant, equilibrium and equivalency conditions derived for the general system of forces apply to coplanar system of forcers as well. These relations, however, can be simplified by realizing that *all components of the system of forces in the z-directions are equal to zero*.

Resultant

(reduction of the system w.r.t. the origin)

$$\vec{F}_r = \sum_{i=1}^{n} \vec{F}_i$$

... Translational effect

$$\begin{cases} F_{rx} = \sum_{i=1}^{n} F_{ix} \\ F_{ry} = \sum_{i=1}^{n} F_{iy} \\ F_{rz} = \sum_{i=1}^{n} F_{iz} = 0 \end{cases}$$

(force acting in the origin O)

$$\vec{M}_O = \sum_{i=1}^{n} \vec{M}_{Oi} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i$$

... Rotational effect

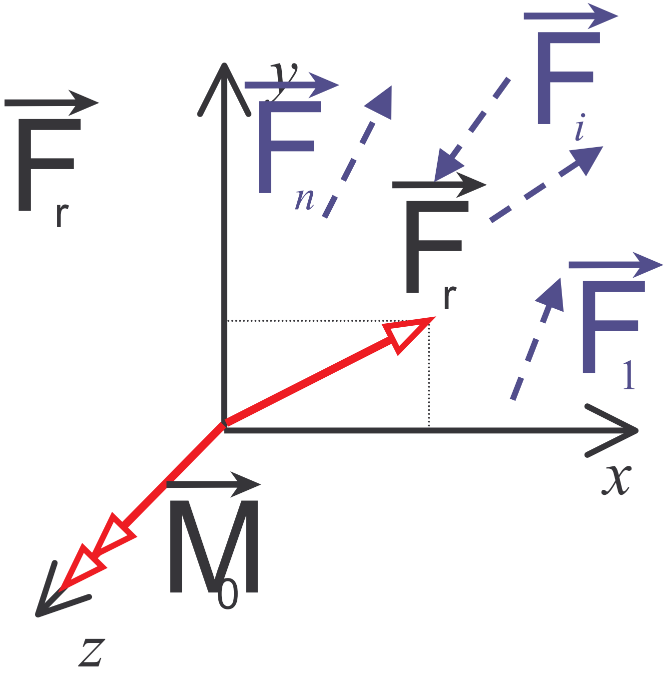
$$\begin{cases} M_x = \sum_{i=1}^{n} M_{ix} = \sum_{i=1}^{n} \left(\cancel{F_{iz}}^{\,0} y_i - F_{iy} \cancel{z_i}^{\,0} \right) = 0 \\ M_y = \sum_{i=1}^{n} M_{iy} = \sum_{i=1}^{n} \left(F_{ix} \cancel{z_i}^{\,0} - \cancel{F_{iz}}^{\,0} x_i \right) = 0 \\ M_z = \sum_{i=1}^{n} M_{iz} = \sum_{i=1}^{n} \left(F_{iy} x_i - F_{ix} y_i \right) \end{cases}$$

The rotational effect is described by a single non-zero component $\vec{M}_0 \parallel M_z$

It holds: $M_O = \pm\sqrt{0 + 0 + M_z^{\,2}} = M_z = \sum_{i=1}^{n} \left(F_{iy} x_i - F_{ix} y_i \right)$

- since $M_x = M_y = 0$, $\vec{M}_0 \perp$ to the plane *x-y*
- since $F_{rz} = 0$, $\vec{F}_r$ is contained by the plane *x-y*

$\Big\}\quad \vec{M}_0 \perp \vec{F}_r$



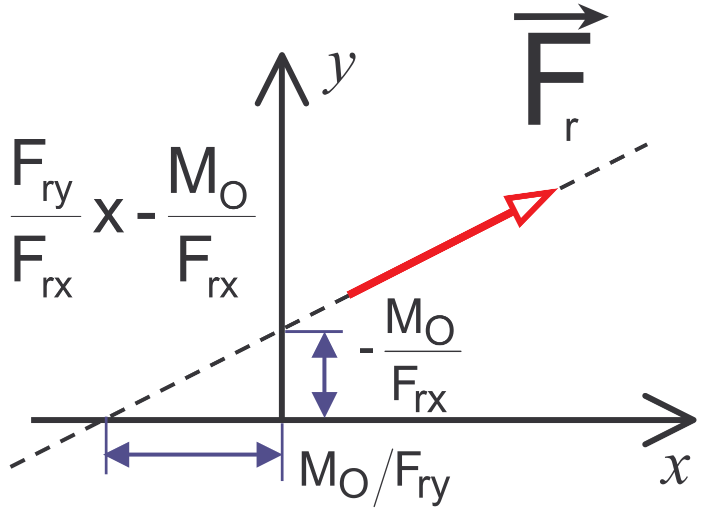
The system of forces can be replaced by a single force $\vec{F}_r$ that complies with the following relations:

$$\vec{F}_r = \sum_{i=1}^{n} \vec{F}_i \qquad \text{a} \qquad M_O = F_{ry}x - F_{rx}y$$

Determines the magnitude, direction and orientation of the resultant:

Determines the line of action of the resultant:

$$y = \frac{F_{ry}}{F_{rx}}x - \frac{M_O}{F_{rx}}$$



Special cases:

- $\vec{F}_r \neq \vec{0}$, $M_O = 0 \Rightarrow$ the resultant is a single force $\vec{F}_r$ acting along the line of action that passes through the c.s. origin

- $\vec{F}_r = \vec{0}$, $M_O \neq 0 \Rightarrow$ the resultant is a force couple acting in the *x-y* plane and giving the moment $M_d = M_O$

- $\vec{F}_r = \vec{0}$, $M_O = 0 \Rightarrow$ the system of forces is in the state of equilibrium

Equations of equilibrium

The system of n forces $\{\vec{F}_i\}$ is in the state of equilibrium,
if their resultant force and resultant moment are both equal to zero:

$$\boxed{\begin{gathered}\sum_{i=1}^{n} F_{ix} = 0 \\ \sum_{i=1}^{n} F_{iy} = 0\end{gathered}}$$

$$\sum_{i=1}^{n} F_{iz} = 0$$

Fulfilled identically

$$\sum_{i=1}^{n} M_{ix} = \sum_{i=1}^{n} \left(F_{iz} y_i - F_{iy} z_i\right) = 0$$

$$\sum_{i=1}^{n} M_{iy} = \sum_{i=1}^{n} \left(F_{ix} z_i - F_{iz} x_i\right) = 0$$

Fulfilled identically

$$\boxed{\sum_{i=1}^{n} M_{iz} = \sum_{i=1}^{n} \left(F_{iy} x_i - F_{ix} y_i\right) = 0}$$

3 conditions

Problem of equilibrium

Consider a system of n forces $\{\vec{F}_i\}$. Bring this system of forces to the state of equilibrium by introducing a new system of m forces $\{\vec{R}_j\}$.

$$\sum_{i=1}^{n} \vec{F}_i + \sum_{j=1}^{m} \vec{R}_j = \vec{0} \qquad \sum_{i=1}^{n} \vec{M}_{OFi} + \sum_{j=1}^{m} \vec{M}_{ORj} = \vec{0}$$

In terms of components:

$$\sum_{i=1}^{n} F_{ix} + \sum_{j=1}^{m} R_{jx} = 0$$

$$\sum_{i=1}^{n} F_{iy} + \sum_{j=1}^{m} R_{jy} = 0$$

$$\sum_{i=1}^{n} M_{OFi} + \sum_{j=1}^{m} M_{ORj} = 0$$

Solvability conditions:

- 3 equations - 3 unknowns
- determinant of the system of equations $\neq 0$

Problem of equivalency

Consider a system of *n* forces $\{\vec{F}_i\}$. Replace this system forces by a new system of *m* forces $\{\vec{R}_j\}$ such that the resulting effect does not change.

$$\sum_{i=1}^{n} \vec{F}_i = \sum_{j=1}^{m} \vec{R}_j \qquad \sum_{i=1}^{n} \vec{M}_{OFi} = \sum_{j=1}^{m} \vec{M}_{ORj}$$

In terms of components:

$$\sum_{i=1}^{n} F_{ix} = \sum_{j=1}^{m} R_{jx}$$

$$\sum_{i=1}^{n} F_{iy} = \sum_{j=1}^{m} R_{jy}$$

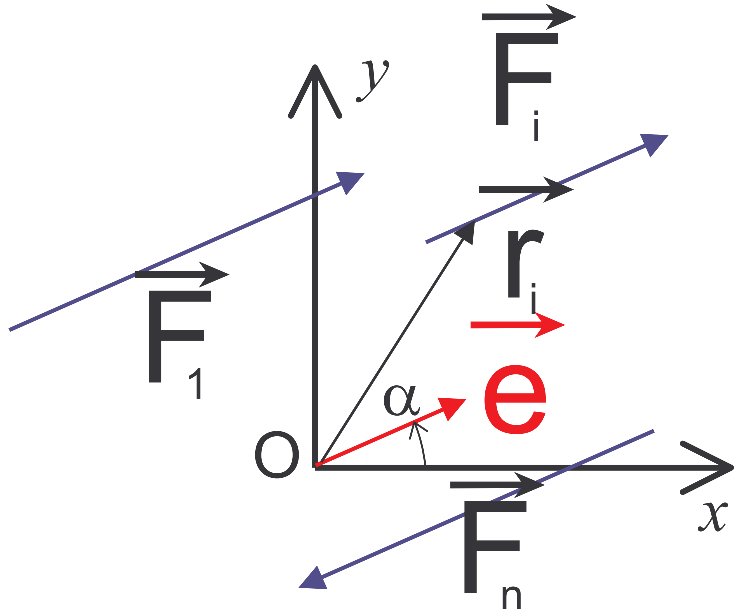
$$\sum_{i=1}^{n} M_{OFi} = \sum_{j=1}^{m} M_{ORj}$$

Solvability conditions:

- 3 equations - 3 unknowns
- determinant of the system of equations $\neq 0$

3.3.4 Coplanar system of parallel forces

A special case of the general system of forces, in which the lines of actions of all forces are mutually parallel and contained by the same plane.



$$\{\vec{F}_i\} = \{F_i \vec{e}\}$$

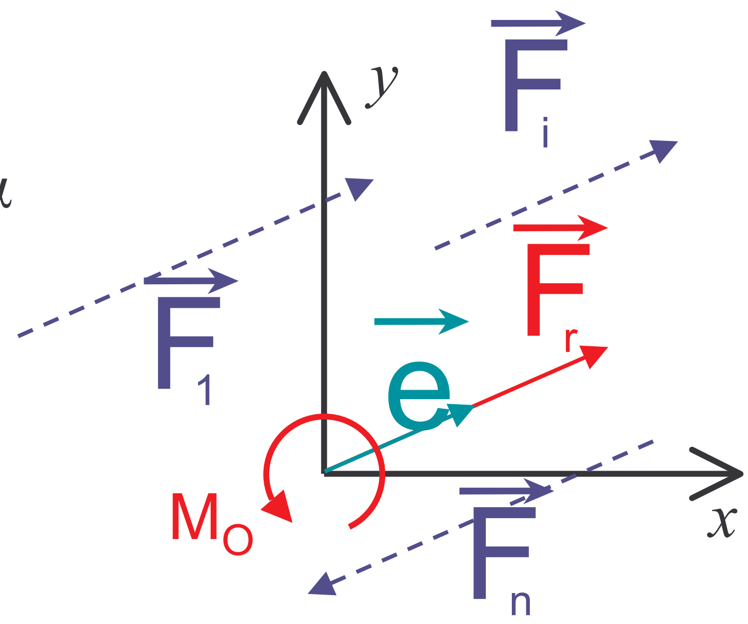
$$\vec{e} = \{e_x, e_y, 0\} = \{\cos\alpha, \sin\alpha, 0\}$$

The resultant

$$\vec{F}_{rx} = \sum_{i=1}^{n} \vec{F}_{ix} = \sum_{i=1}^{n} F_i \cos\alpha = \cos\alpha \sum_{i=1}^{n} F_i = F_r \cos\alpha$$

$$\vec{F}_{ry} = \sum_{i=1}^{n} \vec{F}_{iy} = \sum_{i=1}^{n} F_i \sin\alpha = \sin\alpha \sum_{i=1}^{n} F_i = F_r \sin\alpha$$

$$M_O = \sum_{i=1}^{n} M_{Oi} = \sum_{i=1}^{n} \vec{F}_i (x_i \sin\alpha - y_i \cos\alpha)$$



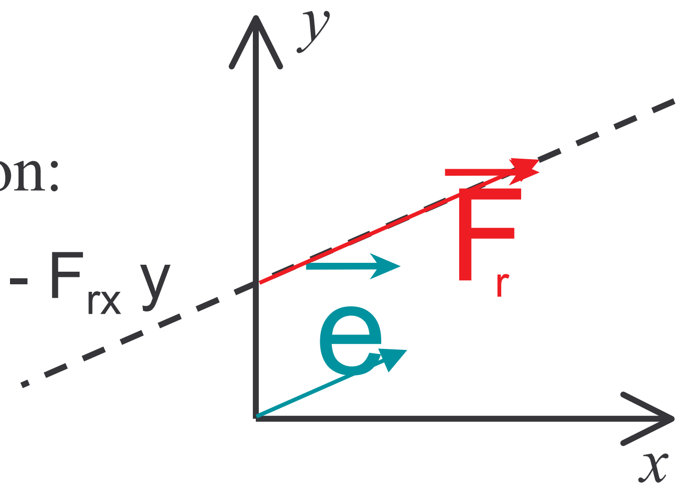
Magnitude:

$$F_r = \sqrt{F_{rx}^2 + F_{ry}^2 + F_{rz}^2} =$$

$$\sqrt{\left(\sum_{i=1}^{n} F_i\right)^2 \left(\cos^2\alpha + \sin^2\alpha + 0\right)} = \sum_{i=1}^{n} F_i$$

Line of action:

$$M_O = F_{ry}\, x - F_{rx}\, y$$



Example: Determine the resultant force and the resultant moment of the coplanar system of parallel forces $\{\vec{F}_i\}$ with their lines of action parallel to the y axis.

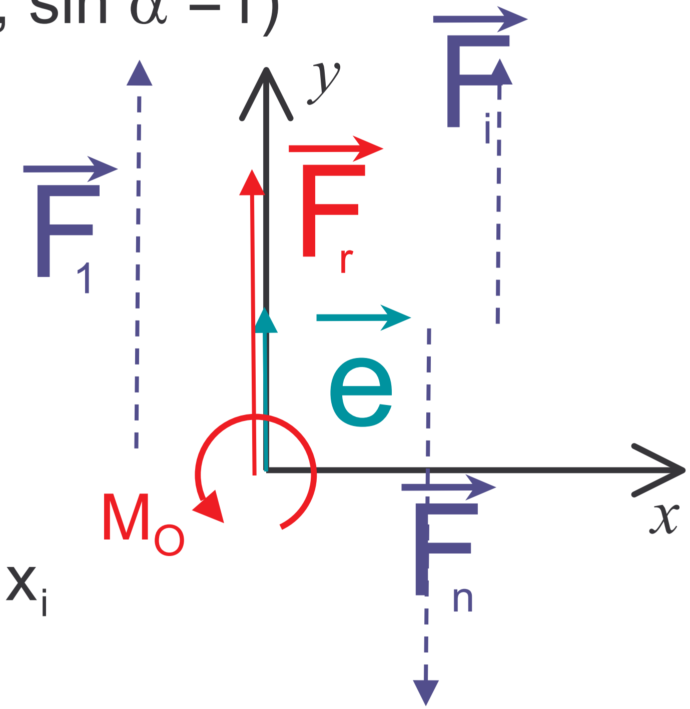
For all forces in the system: $\alpha = \pi/2$ ($\cos\alpha = 0$, $\sin\alpha = 1$)

$\Rightarrow F_{ix} = 0,\ F_{iy} = F_i$

Than: $F_{rx} = 0$

$$F_{ry} = \sum_{i=1}^{n} F_{iy} = \sum_{i=1}^{n} F_i = F_r$$

$$M_O = \sum_{i=1}^{n} M_{Oi} = \sum_{i=1}^{n} F_i\left(x_i \sin\alpha - y_i \cos\alpha\right) = \sum_{i=1}^{n} F_i x_i$$



The line of action of the resultant force:

$M_O = F_{ry}\,x - F_{rx}\,y = F_r\,x$ ... the line || to the y-axis in the distance of $x = M_O / F_r$ measured from the origin O.