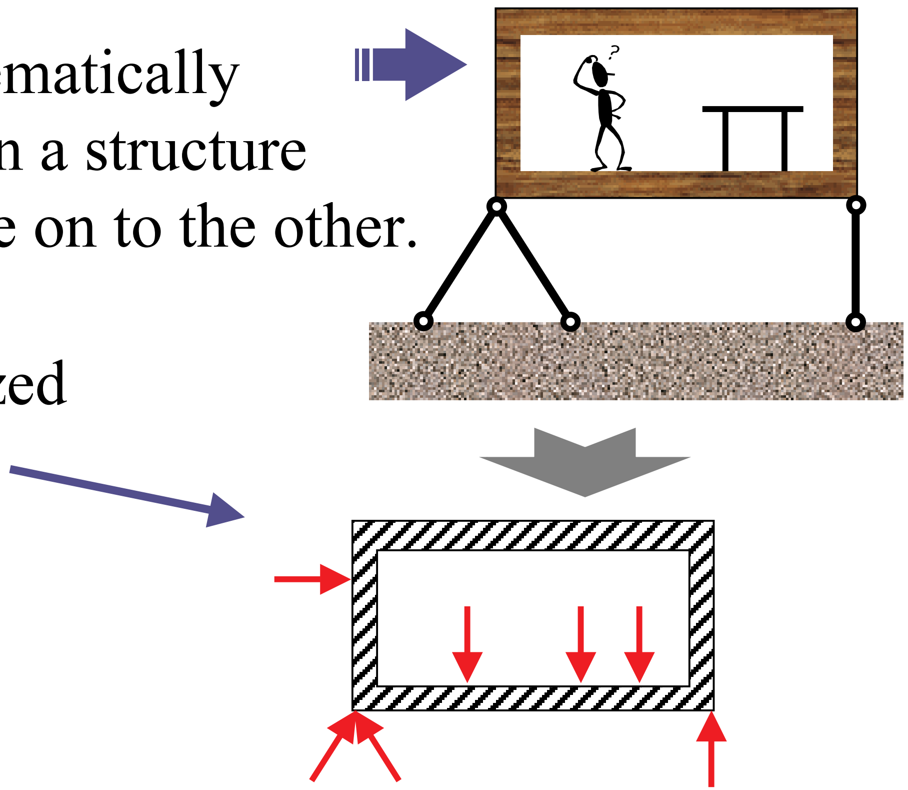
3.2 Forces acting on a solid body

3.2.1 Task, assumptions

The basic task:
In more general way describe mathematically
The mechanical effects of loading on a structure
And effects of one part of a structure on to the other.

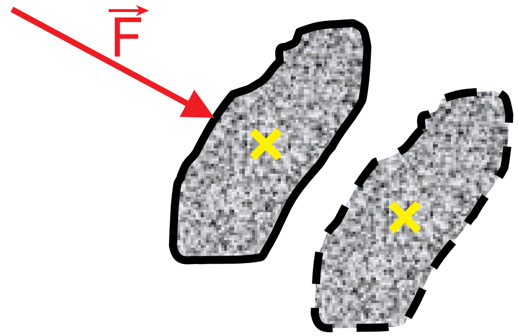


Structure (its parts) will be idealized
By rigid (undeformable) bodies
In 2D (plates) or 3D (solids).

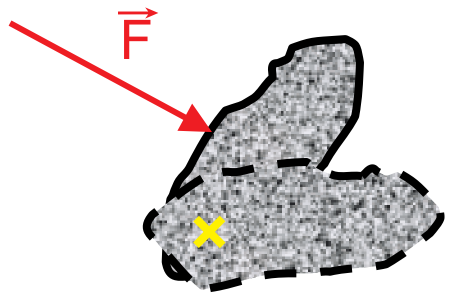
Effects will be described again
Using the vector quantities –
Forces and static moments.

3.2.3 Effect of forces on a rigid body

a) Points on a body found along the line of action of a force are subjected to translation that depends on the force magnitude, direction and orientation.

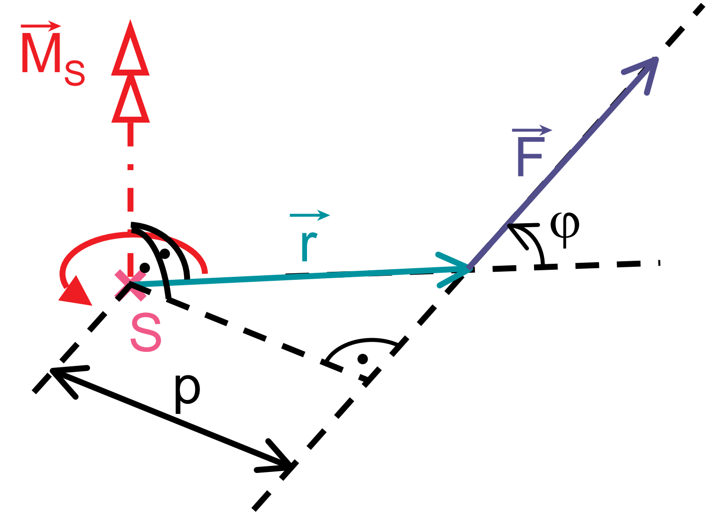


- Points on a body not found along the line of action of a force are subjected to both translation and rotation.



3.2.4 Static moment of a force about a point

The static moment $\vec{M}_S$ of a force about a point is introduced to express the rotational effect of a force $\vec{F}$ about a point S.



$$\vec{M}_S = \vec{r} \times \vec{F}$$

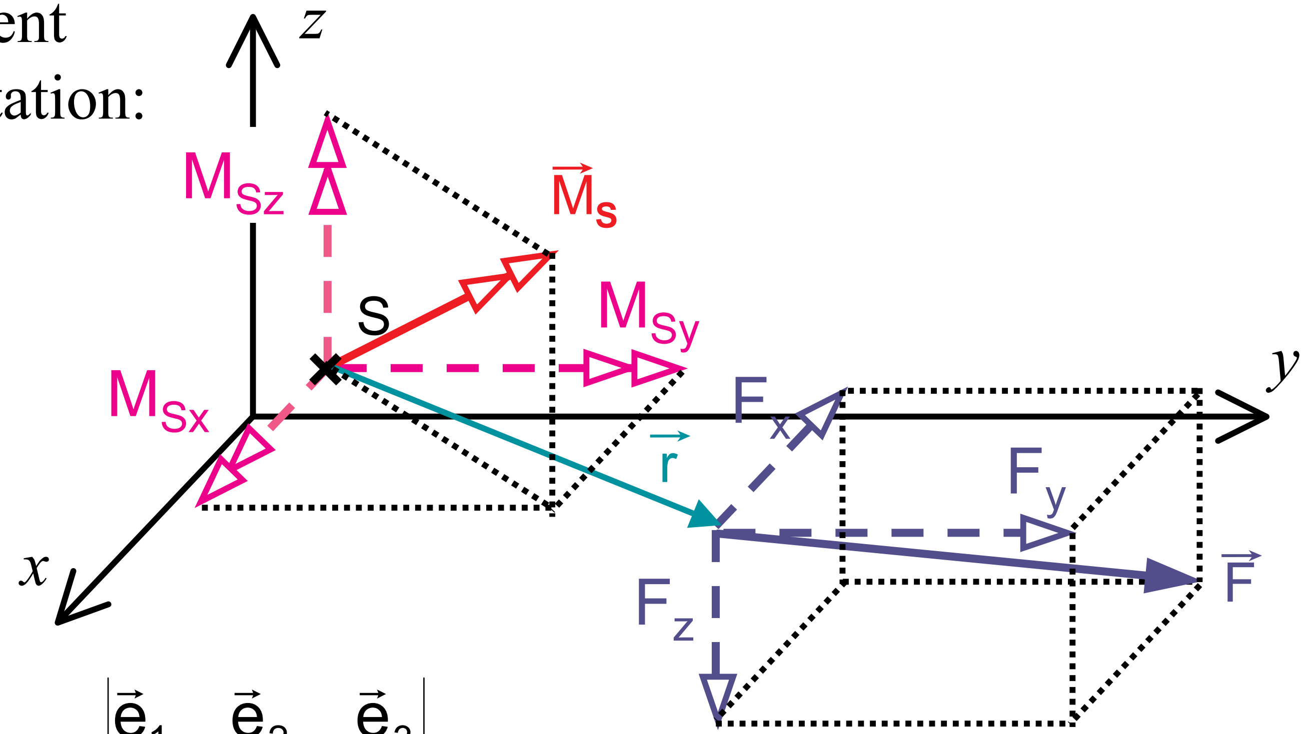
$\vec{r}$ is called the position vector with origin at point S and the end point at any location along the force line of action

From the definition of vector (cross) product:

- $\vec{M}_S$ is a vector tied to point S and perpendicular to the plane specified by point S and force vector $\vec{F}$
- vectors $\vec{r}$, $\vec{F}$ a $\vec{M}_S$ form the right-handed coordinate system
- magnitude $M_S = r\,F \sin\varphi = \pm F\,p$

Basic unit:
Nm (Newton meter)
1 Nm = 1 kg m^2 s^{-2}

Component representation:



$$\vec{M}_s = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(r_y F_z - F_y r_z\right)\vec{e}_1 + \left(r_z F_x - F_z r_x\right)\vec{e}_2 + \left(r_x F_y - F_x r_y\right)\vec{e}_3$$

$$= M_{Sx}\vec{e}_1 + M_{Sy}\vec{e}_2 + M_{Sz}\vec{e}_3 = \{M_{Sx}, M_{Sy}, M_{Sz}\}$$

- magnitude of vecor $\vec{M}_S$:

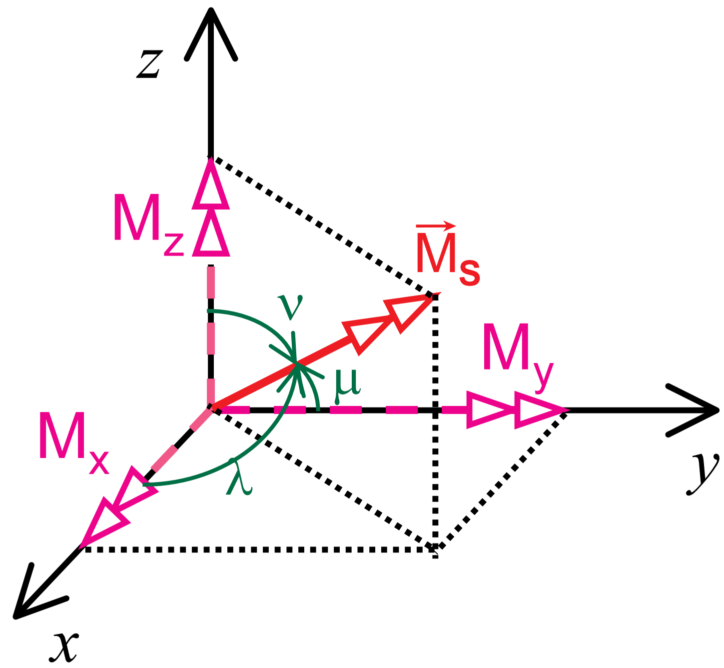
$$M_S = (M_{Sx}^2 + M_{Sy}^2 + M_{Sz}^2)^{1/2}$$

- directional angles:

$$\cos\lambda = \frac{M_{Sx}}{M_S}$$

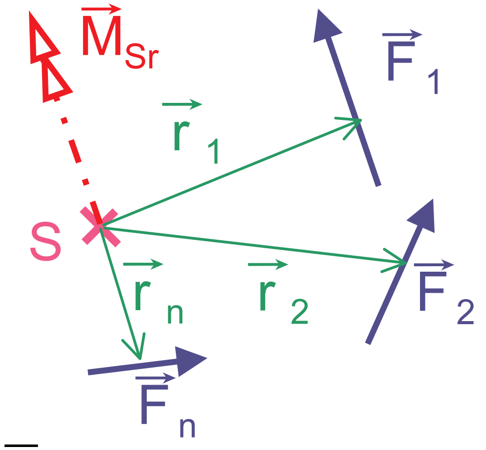
$$\cos\mu = \frac{M_{Sy}}{M_S}$$

$$\cos\nu = \frac{M_{Sz}}{M_S}$$



$$\cos^2\lambda + \cos^2\mu + \cos^2\nu = 1$$

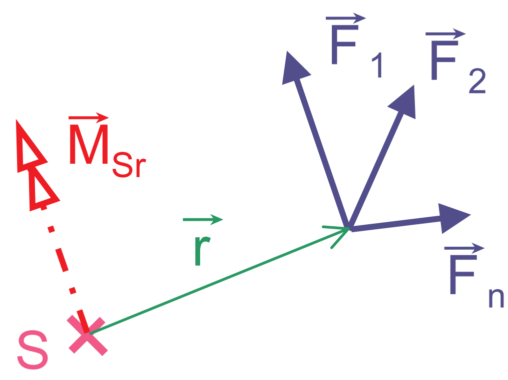
- static moment of a system of forces about a point



$$\vec{M}_{Sr} = \sum_{i=1}^{n} \vec{M}_{Si} = \sum_{i=1}^{n} \left(\vec{r}_i \times \vec{F}_i \right)$$

- static moment of a concurrent system forces – moment (the Varignon) theorem

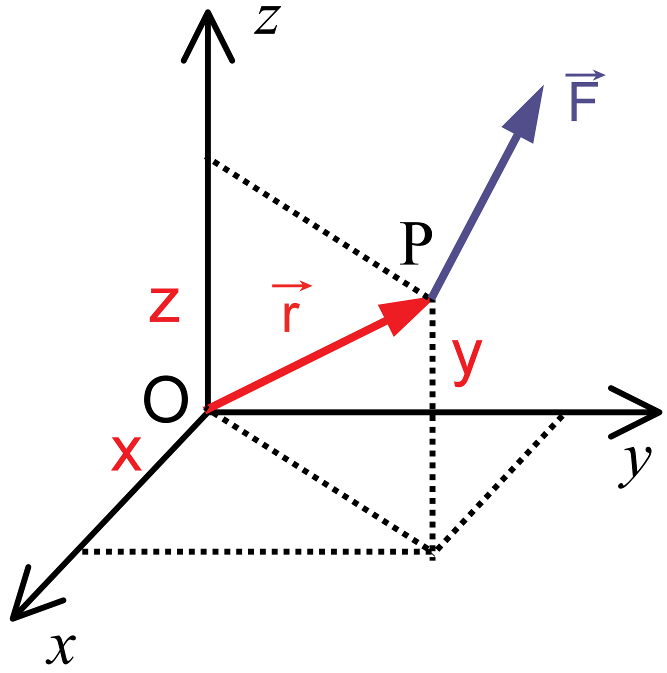
$$\vec{M}_{Sr} = \sum_{i=1}^{n} \vec{M}_{Si} = \sum_{i=1}^{n} \left(\vec{r} \times \vec{F}_i \right) = \vec{r} \times \sum_{i=1}^{n} \vec{F}_i = \vec{r} \times \vec{F}_r$$



Sum of moments of individual forces about point S	=	Moment of the resultant force about point S

Special case: Static moment of a force about the origin

vector $\vec{r} = \{x, y, z\}$ is a position vector containing an an arbitrary point P located on the line of action of $\vec{F}$



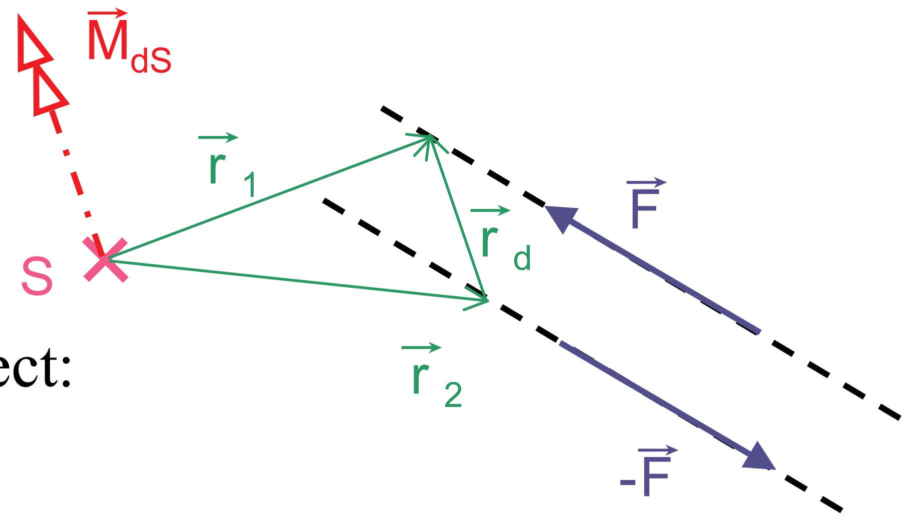
$$\vec{M}_O = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(yF_z - F_y z\right)\vec{e}_1 + \left(zF_x - F_z x\right)\vec{e}_2 + \left(xF_y - F_x y\right)\vec{e}_3$$

$$= M_{Ox}\vec{e}_1 + M_{Oy}\vec{e}_2 + M_{Oz}\vec{e}_3 = \{M_{Ox}, M_{Oy}, M_{Oz}\}$$

3.2.5 Moment of two parallel forces (couple)

- couple forces = two parallel forces having the same magnitude but opposite direction



- force (translational) effect:
  $\vec{F} + (-\vec{F}) = \vec{0}$

- static moment about S:
  $\vec{M}_{dS} = (\vec{r}_1 \times \vec{F}) + [\vec{r}_2 \times (-\vec{F})] = (\vec{r}_1 - \vec{r}_2) \times \vec{F} = \vec{r}_d \times \vec{F}$
  * does not depend on the position of S, the same about every point
  * about a line normal to the plane that contains the couple forces

- vector $\vec{M}_d = \vec{r}_d \times \vec{F}$ is called <u>couple</u>
  - * vector about a line normal to the plane containing the couple forces
  - * represents only rotational effect
  - * magnitude $M_d = r_d\, F \sin\varphi = \pm F\, p$
  - * components:

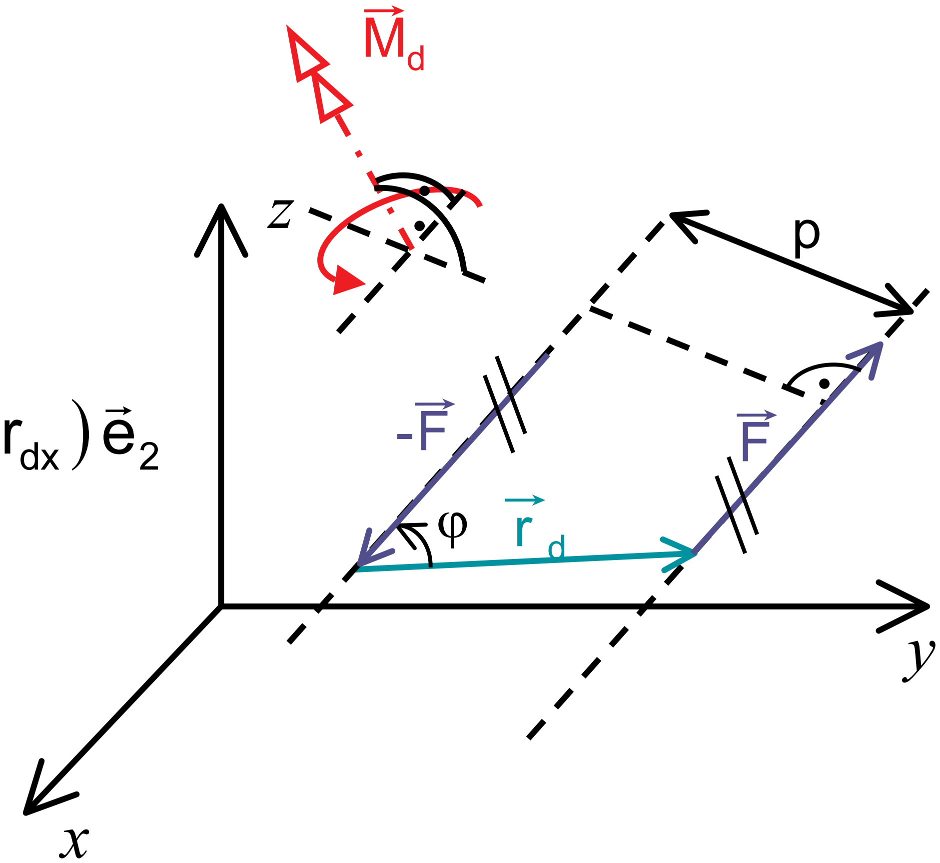
$$\vec{M}_d = \vec{r}_d \times \vec{F} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ r_{dx} & r_{dy} & r_{dz} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(r_{dy}F_z - F_y r_{dz}\right)\vec{e}_1 + \left(r_{dz}F_x - F_z r_{dx}\right)\vec{e}_2$$

$$+ \left(r_{dx}F_y - F_x r_{dy}\right)\vec{e}_3$$

$$= M_{dx}\vec{e}_1 + M_{dy}\vec{e}_2 + M_{dz}\vec{e}_3$$

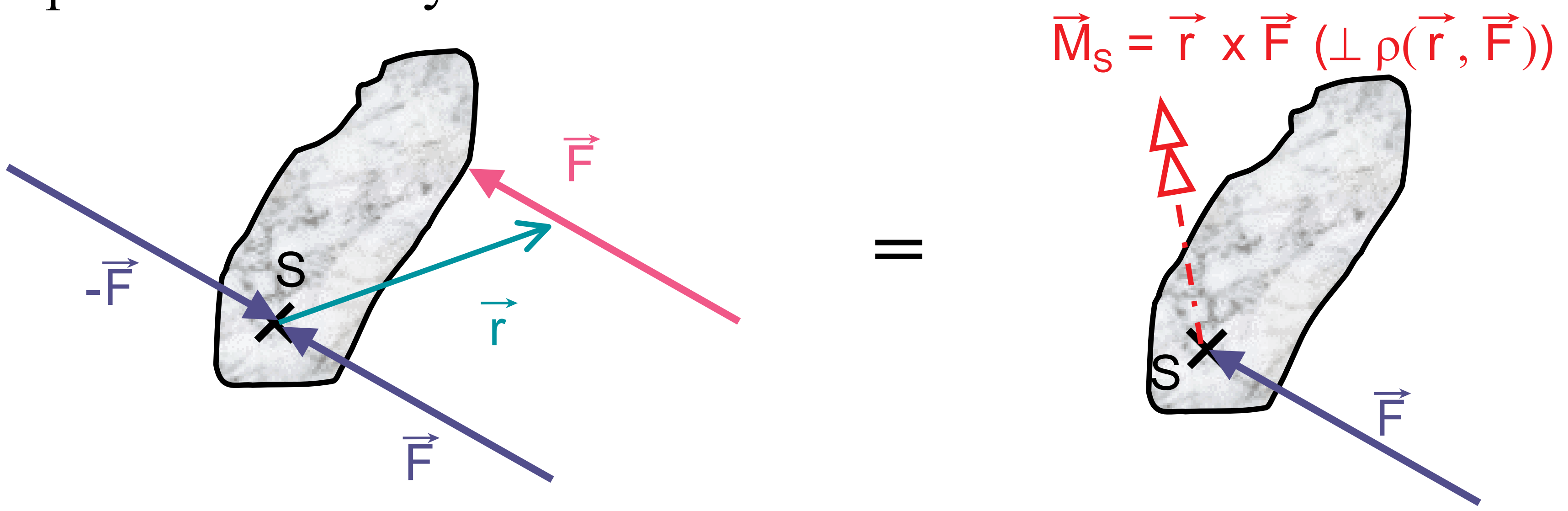
$$= \{M_{dx}, M_{dy}, M_{dz}\}$$



* the couple $\vec{M}_d$ is a free vector, the effect of such a vector is not changed if the couple forces are:
  - rotated or shifted in their own plane
  - placed into a parallel plane
  - replaced by a different couple forces acting in the same or parallel plane and have the same resulting moment $\vec{M}_d$

3.2.6 Reduction of a force to a point

= representation of the static effect of a force $\vec{F}$ on to a given point S in a body.

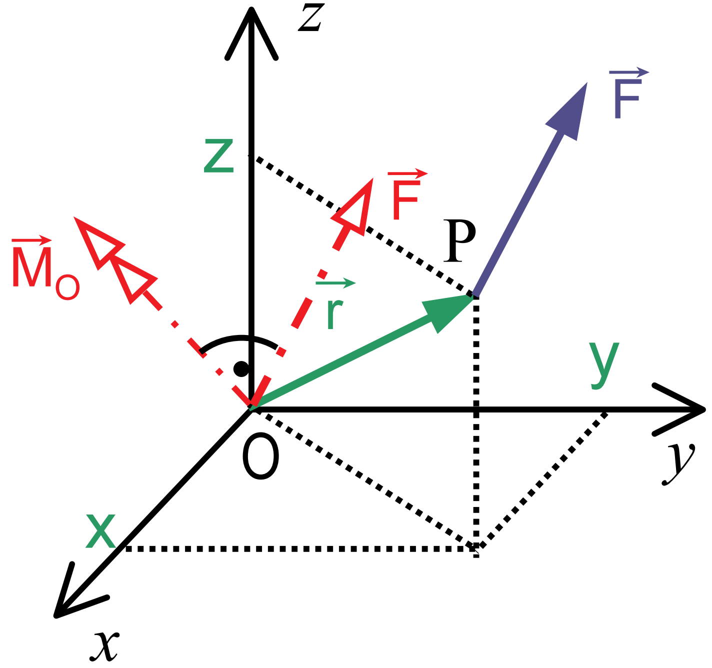


Effect of a force F on a point S:

- translational represented by the force vector $\vec{F}$ acting in S
- rotational given by the moment vector $\vec{M}_S$ of the force $\vec{F}$ about S

Special case: <u>Reduction of a force to the origin</u>

Vector $\vec{r} = \{x, y, z\}$ *is a position vector with end point P resting on the line of action of* $\vec{F}$.



Effect of a force F on the origin:

- <u>translational</u>: $\vec{F} = \{F_x, F_y, F_z\}$
- <u>rotational</u> :

$$\vec{M}_O = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(yF_z - F_y z\right)\vec{e}_1 + \left(zF_x - F_z x\right)\vec{e}_2 + \left(xF_y - F_x y\right)\vec{e}_3$$

$$= M_{Ox}\vec{e}_1 + M_{Oy}\vec{e}_2 + M_{Oz}\vec{e}_3 = \{M_{Ox}, M_{Oy}, M_{Oz}\}$$

Note: $\vec{M}_O \perp \vec{F} \Rightarrow \vec{M}_o \,.\, \vec{F} = M_{Ox}F_x + M_{Oy}F_y + M_{Oz}F_z = 0$

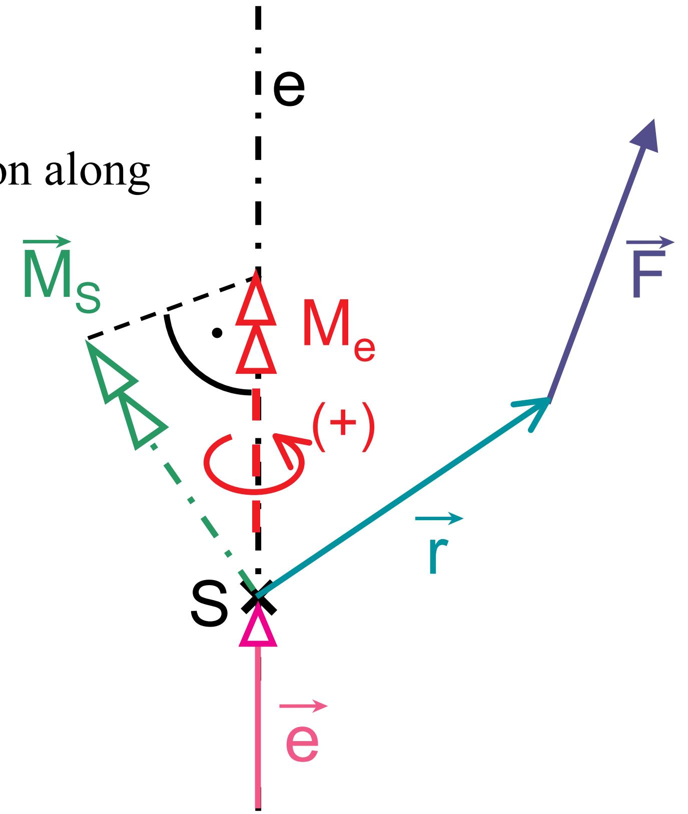
3.2.7 Static moment of a force about a line

- notation M_e
- definition: $M_e = \vec{e} \cdot (\vec{r} \times \vec{F})$
  - * $\vec{e}$ is a unit vector in the direction of e
  - * $\vec{r}$ is a vector with the starting point at any location along e and the end point at any location along the line of action of $\vec{F}$
- <u>scalar</u> representing the rotational effect of $\vec{F}$ about e
- representation:

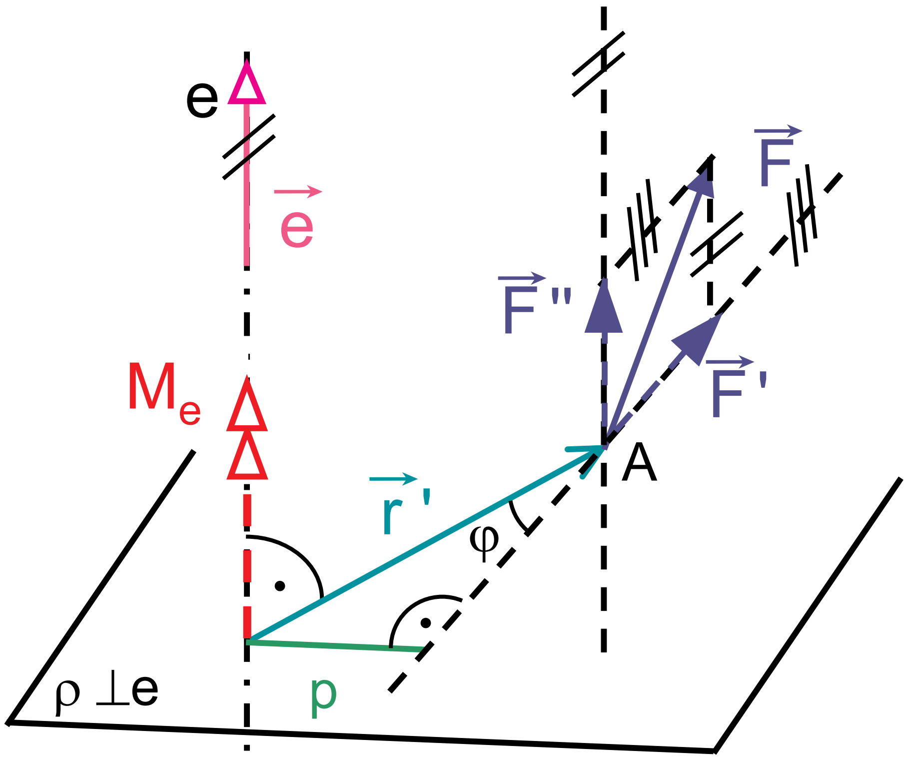
$$M_e = \underbrace{\vec{e} \cdot \underbrace{(\vec{r} \times \vec{F})}}$$

$\vec{M}_S$ ... static moment of $\vec{F}$ about e passing through S

projection $\vec{M}_S$ along e



- another way of representation



- choose $\vec{r}\,' \perp e$
- plane $\rho \perp e$, $A \in \rho$
- $\vec{F}$ represented by the summation

  $\vec{F} = \vec{F}' + \vec{F}''$, where

  $\vec{F}'$ is normal projection of $\vec{F}$ into ρ

  $\vec{F}''$ is normal projection of $\vec{F}$

  along line || with e

$$M_e = \vec{e}\,.\,(\vec{r}\,' \times \vec{F}) = \vec{e}\,.\,[\vec{r}\,' \times (\vec{F}' + \vec{F}'')] = \vec{e}\,.\,\overbrace{(\vec{r}\,' \times \vec{F}')}^{\text{vector } \parallel\, \vec{e}} + \cancel{\vec{e}\,.\,\overbrace{(\vec{r}\,' \times \vec{F}'')}^{\text{vector } \perp\, \vec{e}}}]$$

$$M_e = \vec{e}\,.\,(\vec{e}\; r'\, F' \sin\varphi) = \underline{r'\, F' \sin\varphi = \pm F'\, p}$$

Special case: <u>Static moment about coordinate axes</u>

Axis x: $\vec{e}_1 = \{1,0,0\}$

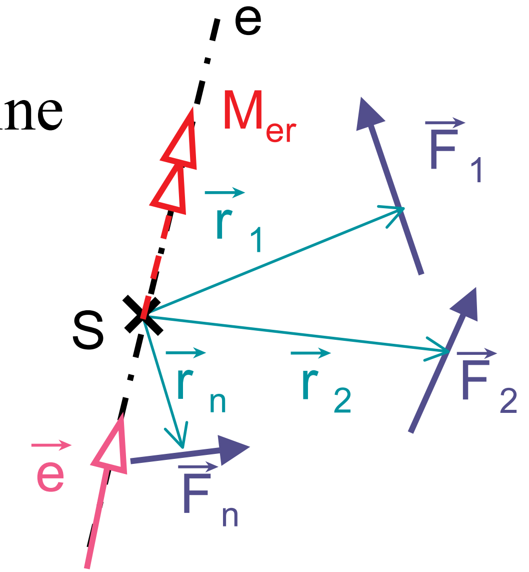
$$M_x = \vec{e}_1 \bullet \left(\vec{r} \times \vec{F}\right) = \begin{vmatrix} 1 & 0 & 0 \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \left(yF_z - F_y z\right)$$

Axis y: $M_y = z\,F_x - x\,F_z$

Axis z: $M_z = x\,F_y - y\,F_x$

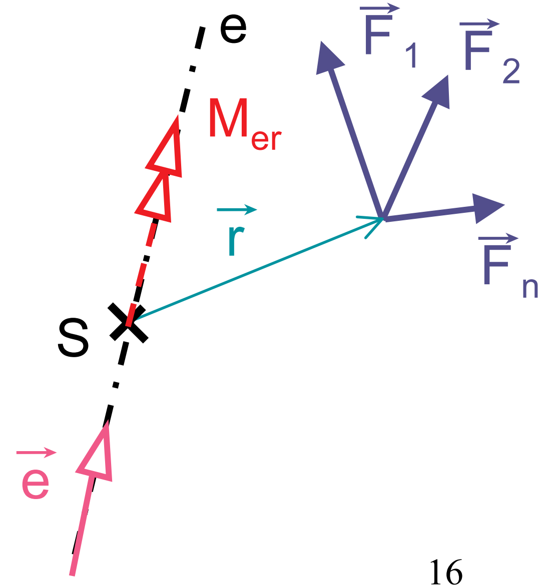
Compare with the components of the static vector about origin!

- static moment of a system of forces about a line



$$M_{er} = \sum_{i=1}^{n} M_{ei} = \sum_{i=1}^{n} \vec{e}_i \cdot \left(\vec{r}_i \times \vec{F}_i\right)$$

- static moment of a concurrent system of forces – moment (the Varignon) theorem



$$M_{er} = \sum_{i=1}^{n} M_{ei} = \vec{e} \cdot \left(\vec{r} \times \sum_{i=1}^{n} \vec{F}_i\right) = \vec{e} \cdot \left(\vec{r} \times \vec{F}_r\right)$$

Sum of moments of individual forces about a line e	=	Moment of the resultant force about a line e

3.2.8 Static moment of couple forces about a line

$$M_{de} = \vec{e} \cdot (\vec{r}_1 \times \vec{F}) + \vec{e} \cdot [\vec{r}_2 \times (-\vec{F})] = \vec{e} \cdot [(\vec{r}_1 - \vec{r}_2) \times \vec{F}]$$

$= \vec{e} \cdot (\vec{r}_d \times \vec{F}) = \vec{e} \cdot \vec{M}_d$ ... projection of moment $\vec{M}_d$ along the axis $\vec{e}$